ÜBERPRO – A SEMINAR CONSTRUCTED TO CONFRONT
THE TRANSITION PROBLEM FROM SCHOOL TO
UNIVERSITY MATHEMATICS, BASED ON
EPISTEMOLOGICAL AND HISTORICAL IDEAS OF
MATHEMATICS¹

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ABSTRACT
In spring 2015 the authors taught an intensive seminar for undergraduate mathematics students, which addressed the transition problem from school to university by bringing to the fore concept changes in mathematical history and the learning biographies of the participants. This article describes how the concepts of empirical and formalistic belief systems can be used to give an explanation for both transitions – from school to university mathematics, and, for secondary mathematics teachers, back to school again. The usefulness of this approach is illustrated by outlining the historical sources and the participants’ activities with these sources on which the seminar is based, as well as some results of the qualitative data gathered during and after the seminar.

Keywords: transition problem, genesis of geometry, secondary school mathematics, higher education, mathematical belief systems.

1. INTRODUCTION TO THE TRANSITION PROBLEM

The transition problem that secondary mathematics teachers experience when moving from school to university (as students), and then again when moving from their university training to teaching mathematics was articulated

¹ “ÜberPro” is an abbreviation of “Übergangsproblematik,” a German word for “transition problem”. With the term “university mathematics” we refer to mathematics courses designed for mathematics students and those pre-service secondary teachers majoring in mathematics (in Germany, these students are usually taught together).
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by Felix Klein (1849-1925) as a “double discontinuity”:

The young university student found himself, at the outset, confronted with problems, which did not suggest, in any particular way, the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honored way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching. (Klein 1908: 1; first author’s translation)

In the following we focus on the “first discontinuity”, referring to the transition from school to university and postulating an epistemological gap between school and university mathematics. We are encouraged by the more than 100-year-old problem, for which definitive solutions do not seem to appear on the horizon (Gueudet 2008). Unfortunately, dropout rates (especially in western countries) remain at a constantly high level. In Germany, approximately 50% of students studying mathematics or mathematics-related fields stop their efforts before finishing a bachelor’s degree (Heublein et al. 2012). In the United States, attrition rates for mathematics majors are differentiated between two undergraduate degrees available – bachelor’s (four-year degree) and associate’s (two-year degree). The National Center for Education Statistics (NCES) reported that for the years 2003 through 2009, 38% of mathematics majors entering university with the intent to earn a bachelor’s degree left the major (Chen 2013). Similarly for those students intending to earn an associate’s degree, some 78% left the major. This leads again to an (at least perceived) intensification of research in this field.

Furthermore, recent investigations in the United States have focused on the critical role that success in calculus course taking plays in undergraduate students’ ambition for and persistence in mathematics. To date, many of the resulting publications from the Mathematical Association of America National Study of Calculus have highlighted the importance of student attributes on their success (e.g., Bressoud et al. 2013); however, identifying concrete ways in which students may be successful in negotiating the transition from secondary school mathematics student to first-year university mathematics student is absent from the literature.

In 2011, the most important professional associations regarding mathematics (education) in Germany (DMV-Mathematics, GDM-Mathematics Education, and MNU-STEM Education) formed a task force regarding the problem of transition (cf. http://www.mathematik-schule-hochschule.de). Then, in February 2013, a scientific conference with the topic “Mathematik im Übergang Schule/Hochschule und im ersten Studienjahr” (“Mathematics at the Crossover School/University in the First Academic Year”) in Paderborn, Germany
attracted almost 300 participants giving over 80 talks regarding the problematic transition process from school to university mathematics. The proceedings of this conference (Hoppenbrock et al. 2013) and its predecessor on special transition courses (Biehler et al. 2014) give an impressive overview on the necessity and variety of approaches regarding this matter. Interestingly a vast majority of the studies and best practice examples for “transition courses” locate the problem in the context of deficits (going back as far as junior high school) regarding the content knowledge of freshmen at universities.

In the “pre-course and transition course community” it seems to be consensus by now that existing deficits in central fields of lower-secondary schools’ mathematics make it difficult for freshmen to acquire concepts of advanced elementary mathematics and to apply these. Fractional arithmetic, manipulation of terms or concepts of variables have an important role, e.g., regarding differential and integral calculus or non-trivial application contexts and constitute insuperable obstacles if not proficiently available. (Biehler et al. 2014: 2; first author’s translation)

The question of how to provide first semester university students with obviously lacking content knowledge is certainly an important facet of the transition problem. However, as the results of a recent empirical study suggest, there are other, deeper problem dimensions that aid in further understanding the issue.

2. MOTIVATION FOR DEVELOPING THE SEMINAR

To investigate new perspectives on the transition problem, approximately 250 pre-service secondary school teachers from the University of Siegen and the University of Cologne in 2013 were asked for retrospective views on their way from school to university mathematics. When the survey was disseminated, the students had been at the universities for about one year. Surprisingly, the systematic qualitative content analysis of the data (Huberman & Miles 1994, Mayring 2002) showed that from the students’ point of view it was not the deficits in content knowledge that dominated their description of their own way from school to university mathematics. Instead, students articulated a feeling of “differentness” between school and university mathematics that did not relate simply to a rise in content-specific requirements. To illustrate this point of “differentness” we selected two exemplar responses from the questionnaire responses to the question,

**What is the biggest difference or similarity between school and university mathematics? What prevails? Explain your answer.**

Student (male, 19 years): “The fundamental difference develops as mathematics in school is taught “anschaulich”[1], whereas at university there is a rigid modern-axiomatic structure characterizing mathematics. In general there are more differences than similarities, caused by differing aims” (first author’s translation)

At this first student’s point we can only speculate on the term “aims”, but in
reference to other formulations in his survey responses it seems possible that he distinguished between general education (in German, “Allgemeinbildung”) as an aim for school and specialized scientific teacher-training at universities.

The second example is impressive in the same sense:

Student (female, 20 years):

Figure 1. A student’s articulation of difference or similarity between school and university mathematics.

In many cases the students clearly distinguished between school and university mathematics, which is most prominent in the second example. For this student, school mathematics and university mathematics are so different, that the only remaining similarity (in German, “Gemeinsamkeit”) is the word “mathematics”. This “differentness” encountered by the students is specified in further parts of the questionnaire with terms as vividness, references to everyday life, applicability to the real world, ways of argumentation, mathematical rigor, axiomatic design, etc.²

Using additional results of studies with a similar interest (e.g., Gruenwald et al. 2004, Hoyles et al., 2001) we arrived at the preliminary conclusion that pre-service mathematics teachers clearly distinguish between school and university mathematics with regard to the nature of mathematics. In the terms of Hefendehl-Hebeker Ableitinger and Herrmann, the students encounter an “Abstraction shock” (Hefendehl-Hebeker et al. 2010), meaning that students have serious difficulties regarding a dramatically increased level of abstraction at the beginning of their undergraduate courses in mathematics. Schichel and Steinbauer (2009: 1; first author’s translation) describe the same phenomenon, when saying that,

² The cited study has not been published in total so far. However, partial results have been published in Witzke 2013a, 2013b.
Abstraction shock: The level of abstraction regarding the teaching of university mathematics is in marked contrast to the teaching in school, where mathematical content is in principal developed on the basis of [concrete] examples. Many students get already lost in the “definition-theorem-proof-jungle” in the first weeks of their university career being faced with an uncommented abstract approach.

To describe and face this problem, we established a framework for further research concerning the transition problem. In the next section, we reconstruct the nature of mathematics communicated explicitly and implicitly in high school and university textbooks, lecture notes, standards, etc., with a special focus on differences to identify in detail what constitutes the abstraction shock described in literature and by students. Thereby we follow the paradigm of constructivism in mathematics education, believing that students construct their own view on mathematics when working and interacting in classroom or lecture hall with the material, problems, and stimulations that course instructors (and students’ peers) provide (Anderson et al. 2000, Bauersfeld 1992).

3. BELIEFS ON MATHEMATICS: TODAY AND IN HISTORY

3.1 Beliefs describing the notion of mathematical objects and activities

The terms nature of mathematics and belief system regarding mathematics are closely linked to each other if we understand learning in a constructive way. Schoenfeld (1985) successfully showed that personal belief systems matter when learning and teaching mathematics:

One’s beliefs about mathematics […] determine how one chooses to approach a problem, which techniques will be used or avoided, how long and how hard one will work on it, and so on. The belief system establishes the context within which we operate […] (Schoenfeld 1985: 45)

From an educational point of view beliefs about mathematics are decisive for our mathematical behavior as the empirical studies of Schoenfeld have shown; the beliefs system was identified as the critical factor determining success in concrete problem solving contexts. Furthermore, prominent among research on beliefs are four categories of beliefs concerning mathematics, which were distinguished by Grigutsch, Raatz and Törner (1998) as aspects: the toolbox aspect, the system aspect, the process aspect and the utility aspect. Liljedahl, Rolka and Roesken (2007) specified this wide range of possible aspects of a mathematical worldview as follows:

In the “toolbox aspect”, mathematics is seen as a set of rules, formulae, skills and procedures, while mathematical activity means calculating as well as using rules, procedures and formulae. In the “system aspect”, mathematics is characterized by logic, rigorous proofs, exact definitions and a precise mathematical language, and doing mathematics consists of
accurate proofs as well as of the use of a precise and rigorous language. In the “process aspect”, mathematics is considered as a constructive process where relations between different notions and sentences play an important role. Here the mathematical activity involves creative steps, such as generating rules and formulae, thereby inventing or re-inventing the mathematics. Besides these standard perspectives on mathematical beliefs, a further important component is the usefulness, or utility [aspect], of mathematics. (Liljedahl et al. 2007: 279)

Very often these beliefs are located within certain fields of tension (in German, “Spannungsfelder”). There is, for example, the process aspect, which is always implicitly connected to its opposite pole the product aspect. Another pair of concepts in this sense is certainly an intuitive aspect on the one hand and a formal aspect on the other, having even a historical dimension: “There is a problem that goes through the history of calculus: the tension between the intuitive and the formal” (Moreno-Armella 2014: 621). These fields of tension may help to describe the problems the students encounter on their way to university mathematics. Especially helpful when looking at the results of the aforementioned survey, representing one important facet, seems to be the tension between what Schoenfeld called an empirical belief [2] system and a formalistic belief system [3] – a convincing analytical distinction following the works of Burscheid and Struve (2010).

The empirical belief system [2] on the one hand describes a set of beliefs in which mathematics is understood as an experimental natural science, which includes deductive reasoning about empirical objects. Struve (1990) and Schoenfeld (1985) have reconstructed this belief system in school, investigating school textbooks and students’ behavior.

Good examples for comparable belief systems, regarding the understanding of mathematics in an empirical way, can be found in the history of mathematics. The famous mathematician Moritz Pasch (1843-1930), who completed Euclid’s axiomatic system, explicitly understood geometry in this way:

The geometrical concepts constitute a subgroup within those concepts describing the real world […] whereas we see geometry as nothing more than a part of the natural sciences. (Pasch 1882: 3)

Thus, mathematics in this sense is understood as an empirical, natural science. This, of course, implies the importance of inductive elements as well as a notion of truth bonded to the correct explanation of physical reality. In Pasch’s examples, Euclidean geometry is understood as a science describing our physical space by starting with evident axioms. Geometry then follows a deductive buildup, but it is legitimized by the power to describe the physical space around us correctly. This understanding of mathematics as an empirical science (on an epistemological level) can be found throughout the history of mathematics, and prominent examples for this understanding are found in many scientists of the 17th and 18th centuries. For example, Leibniz conducted analysis on an empirical level; the objects of his calculus differentialis and calculus
integralis were curves given by construction on a piece of paper and not as today’s abstract functions (cf. Witzke 2009).

The formalistic belief system, on the other hand, describes a set of beliefs in which mathematics is understood as a system of un-interpreted concepts and their connections in propositional functions (in German, “Aussageformen”), which can be established using axioms, (implicit) definitions, and proofs. Davis and Hersh (1981) and Schoenfeld (1985) have reconstructed this belief system as a typical one for professional mathematicians.

Good examples for comparable belief systems, regarding the understanding of mathematics in a formalistic way, can be found in the history of mathematics. The famous mathematician David Hilbert (1862-1943), released geometry completely from any empirically bonded entities:

Whereas Pasch was anxious to derive his fundamental notions from experience and to postulate no more than experience seems to grant. Hilbert started ‘Wir denken uns…’ we imagine three kinds of things… called points… called lines… called planes… we imagine points, lines, and planes in some relations… called lying on, between, parallel, congruent…” (-) “Wir denken uns…” – the bond with reality is cut. Geometry has become pure mathematics. The question of whether and how to apply it to reality is the same in geometry as it is in other branches of mathematics. Axioms are not evident truths. They are not truths at all in the usual sense. (Freudenthal 1961: 14; English translation in Streefland 1993)

Mathematics in this sense can be understood as the formal science. This implies the importance of deductive elements as well as a notion of truth in the sense of logical consistency. This understanding of mathematics as a formal science (on an epistemological level) can be found throughout the history of mathematics after Hilbert. Prominent examples for this understanding are found in many mathematicians of 19th, 20th, and 21st centuries. For example, Kolmogoroff formalized probability theory in this way; the concepts of his Grundbegriffe der Wahrscheinlichkeitsrechnung are sets and measures given by definition in his famous axioms. (cf. Kolmogorov 1973).

So, what is the connection among these elements, mathematics students, and the transition problem? If we examine current textbooks for school mathematics, we see that students at school are likely to acquire an empirical belief system. And, if we examine current course textbooks for university mathematics, we see that students at university are, in contrast, faced with a formalistic belief system (cf. Burscheid & Struve 2009, Schoenfeld 1985, Schoenfeld 2011, Struve 1990, Tall 20133). On epistemological grounds both show parallels to specific historical understandings of mathematics. These

3 In his foundational work, “How humans learn to think mathematically”, David Tall (2013) emphasized an equivalent to Struve’s and Schoenfeld’s empirical belief system when referring to a blend of “Embodiment and Symbolism” prevailing in school. He distinguished this, what he calls worlds of mathematics, from a world of “(Axiomatic) formalism” realized at university level and associated Hilbert - which is quite similar to Burscheid and Struve’s formalistic belief system.
epistemological parallels were fundamental for the design of our “transition problem” seminar for students. The main idea is that the recognition and appreciation of different natures of mathematics in history (i.e. those held by expert mathematicians) can help students to become aware of their own belief system and may guide them to make necessary changes.

4. A DEEPER LOOK INTO SCHOOL AND UNIVERSITY MATHEMATICS

The most recent National Council of Teachers of Mathematics (NCTM) standards (2000) and prominent school textbooks indicate that, for good reasons (cf. the EIS-principle by Bruner (1966), the basic experiences (“Grundерfahrungen”) of Winter (1996) or the three worlds of mathematics by Tall (2013)), mathematics is taught in the context of concrete (physical) objects at school. For example, the NCTM process standards, and in particular “connections” and “representations,” (which are comparable to similar mathematics standards in Germany), focus on empirical aspects of mathematics. At school and in their future career it is important that students “recognize and apply mathematics in contexts outside of mathematics” or “use representations to model and interpret physical, social, and mathematical things” (NCTM 2000: 67). The prominent place of illustrative material and visual representations in the mathematics classroom has important consequences for the students’ views about the nature of mathematics. As we previously mentioned, Schoenfeld (1985, 2011) and Struve (1990, 2010) proposed that students acquire an empiricist belief system of mathematics at school. This is likely to be caused by the fact that mathematics in modern classrooms does not describe abstract entities of a formalistic theory but a universe of discourse ontologically bounded to “real objects”. For example, Probability Theory is bounded to random experiments from everyday life, Fractional Arithmetic to “pizza models”, Geometry to straightedge and compass constructions, Analytical Geometry to vectors as arrows, Calculus to functions as curves (graphs), and so forth.

However, at university things can look totally different. Authors of prominent textbooks (in Germany, as well as in the United States) for beginners at university level depict mathematics in quite a formalistic, rigorous way. For example, in the preface of Abbott’s popular book for undergraduate students, Understanding Analysis, it becomes very clear how mathematicians consider a major difference between school and university mathematics: “Having seen mainly graphical, numerical, or intuitive arguments, students need to learn what constitutes a rigorous proof and how to write one” (Abbott 2001: vi). This view is also transported by Heuser’s popular analysis textbook for first semester students (Heuser 2009: 12; first author’s translation):

The beginner at first feels […] uncomfortable […] with what constitutes mathematics:
- The brightness and rigidity in concept formation
- The pedantic accurateness when working with definitions
- The rigor of proofs which are to be conducted [...] only with the means of logic not with Anschauung. [1]
- Finally the abstract nature of mathematical objects, which we cannot see, hear, taste or smell. [...] This does not mean that there are no pictures or physical applications in Abbott’s book; it is common sense that modern mathematicians work with pictures, figural mental representations, and models. However, in contrast to many students, it is clear to them that these are illustrations or visualizations only, displaying certain logical aspects of mathematical objects (and their relations to others) but by no means representing the mathematical objects in total. This distinction is more explicit if we look at a textbook example. In school textbooks (in Germany) the reference objects for functions are mainly drawn curves. Functions may then virtually be identified with these empirically given curves (Witzke 2014). Tietze, Klika and Wolpers (2000: 72) discussed this context of an analysis like “elementary algebra combined with the sketching of graphs”. Consequently, school textbook authors work with the so-called concept of graphical derivatives (firmly anchored in the curricula) in the context of analysis (see Fig. 2). At university, curves are by no means the reference objects; here they are only one possible interpretation of the abstract notion of function. The graph of a function in formalistic [3] university mathematics is actually only a set of (ordered) pairs.

If we contrast the empirical belief system many students acquire in classroom with the formalistic belief system students are faced with at university we have a model that explains why challenging the transition problem regarding belief systems is necessary for the professionalization of mathematicians and math teachers. For example, in this model the notion of proof differs substantially in school and university mathematics. Whereas at universities (especially in pure mathematics) only formal deductive reasoning is an acceptable method, non-rigorous proofs relying on “graphical, numerical and intuitive arguments” are an essential part of proofs in school mathematics where we explain phenomena of the “real world”. Using Sierpinska’s (1987, 1992) terminology, students in this transition-phase have to overcome a variety of “epistemological obstacles”4, requiring a significant change in their understanding of what mathematics is about.

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4 Following the definition and common usage of the term “epistemological obstacle” in mathematics education, we mean content-based obstacles that are likely to occur in every learning biography, and whose overcoming will eventually lead to a decisive process of cognition. Note that these are referred to as being based in the nature of things in principle and not in the lack of individual cognitive development (cf. Schneider 2014: 214-217).


5. SEMINAR DESIGN, CONTENT, AND IMPLEMENTATION

The findings of the initial questionnaire and the identification of the theoretical considerations, which were described in the preceding paragraphs, were essential for designing a seminar to address the transition problem. The overall aim of the seminar course was to make students aware and to lead them to an understanding of crucial changes regarding the nature of mathematics from school to university, by discussing transcripts, textbooks, standards, historical sources, etc. The different “natures” of mathematics in school and university can also, on an epistemological level, be found in the history of mathematics, as we previously stated. Thus, an understanding of how and why this change (from empirical-physical to formalistic-abstract) took place should be achieved by an historical-philosophical analysis (cf. Davies 2010). This, in fact, is the key notion of the seminar. Thereby we hoped that the students were able to relate their own learning biographies to the historical development of mathematics. This conceptual design of the seminar draws upon positive experience with explicit approaches regarding changes in the belief system of students from science education (esp. “Nature of Science” cf. Abd-El-Khalick & Lederman 2001).

The undergraduate ÜberPro seminar that is the focus of what follows, was designed for students to cope with the transition problem. It was implemented for the first time in February 2015 and was an intensive experience that took place over three days (approximately 18 hours of instruction). Twenty (8 male; 12 female) undergraduate mathematics students, who were also preparing to teach secondary mathematics, participated in the seminar. Table 1 presents the distribution of student age and semester at university of the seminar participants.

Figure 2. Graphical derivatives in a German school textbook (EdM 2010: 203).

4.4.2 Untersuchung auf Monotonie und Extrema mithilfe der 1. Ableitung

| Untersucht, wie das Monotonieverhalten einer Funktion und die Steigungen ihres Graphen zusammenhängen und wie man damit Extremstellen ermitteln kann. |
| a) Jeder zeichnet einen Funktionsgraphen, zu dem sein Partner den Ableitungsgraph skizziert. |
| b) Jeder zeichnet den Graphen einer Ableitung, zu dem sein Partner einen passenden Funktionsgraphen skizziert. |


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The three-day seminar was organized in four parts:

1) Raise attention to the importance of beliefs about and philosophies of mathematics.

2) Historical case study: Geometry from Euclid to Hilbert. (In particular, which questions led to the modern understanding of mathematics?)

3) Exploration of Hilbert’s approach (Or, what characterizes modern formalistic mathematics?)

4) Summary discussion and reflection.

We employed several instructional techniques during the intensive seminar. During the 18 hours of instruction students engaged in small group work, which included engaging in active learning tasks and short discussions, and whole class discussions, which included individual students and small groups sharing their work. The self-activating sequences were enriched by short instructional lectures of the participating mathematics educators (i.e. the first, second, and fourth authors). Moreover, seminar participants worked with a variety of materials, including reading original historical sources, excerpts from research literature, and school textbooks; using hands-on materials to model concepts from projective and hyperbolic geometry; and investigating concepts using dynamic geometry software. We provide further context and description for a number of the seminar activities within the elaboration of the four parts of the seminar that follows.

1) Raise attention to the importance of beliefs about and philosophies of mathematics.

In the first part of the seminar we wanted to make students aware of the idea of different belief systems and natures of mathematics. Here we began with individual reflections and work with authentic material such as transcripts...
from Schoenfeld’s (1985) research that clearly showed the meaning and relevance of the concept of an empirical belief system. Afterwards students compared different types of textbooks: university course textbooks, school textbooks, and historical textbooks.

The three excerpts (Fig. 3) illustrate how we worked within this comparative activity. In the upper left-hand corner of Fig. 3 is a formal university textbook definition of differentiation. It is characterized by a high degree of formalization: the objects of interest are functions defined on real numbers or complex numbers. The excerpt exhibits a highly symbolic definition where the theoretical concept of limit is necessary. In contrast, we see just below an excerpt from a popular German school textbook. Here, the derivative function is defined on a purely empirical level; the upper curve is virtually identified with the term function. Characteristic points are determined by an act of empirical measuring and the slopes of the triangles are then plotted underneath and results in the second graphed curve (graphical derivation).

**Figure 3. Three excerpts of different textbooks for comparison.** University course textbook “Königsberger 2001: 34” (top left), school textbook “Lambacher Schweizer 2009: 55” (bottom left), historical text “Leibniz, Acta Eruditorum,” 1693 (right).

Finally, if we look back to Leibniz (one of the fathers of analysis), with his *calculus differentialis* and *integralis*, we find that he conducted mathematics in a rather empirical way as well (cf. Witzke 2009). His objects were curves given by construction on a piece of paper – properties like differentiability or continuity...
were read out of the curve. Furthermore, there seem to be parallels on an epistemological level between school analysis and historical analysis. For example, Leibniz presented (published in 1693) the invention of the so-called “integrator” (right-hand side of Fig. 3), a machine that was designed to draw an anti-derivative curve by retracing a given curve. So here, as in the school textbook, the empirical objects form the basis of the theory; even more the processes regarding Leibniz’ integrator and the textbooks’ graphical derivation.

During the seminar course, students shared their response to the question, “What is mathematics?” – which were then organized according to the scheme aspect, formalism aspect, process aspect, and utility aspect, similar to those introduced in the items by Grigutsch, Raatz and Törner (1998).

2) Historical case study: Geometry from Euclid to Hilbert. (In particular, which questions lead to the modern formalistic understanding of mathematics?)

An adequate description of the development of the nature of mathematics in the course of history requires more than one book. We referenced the following ones: Bonola (1955) for a detailed historical presentation; Garbe (2001), Greenberg (2004), and Trudeau (1995) for a lengthy historical and philosophical discussion; Ewald (1971), Hartshorne (2000), and Struve and Struve (2010) for a modern mathematical presentation. Additionally, Davis and Hersh (1981) and Davis, Hersh and Marchiotto (1995) presented aspects of the historical and philosophical discussion in a concise manner, and for students, in a relatively easy and accessible way.

The overall aim of the historical case study was to make students aware of how the nature of mathematics changed over history. Regarding our theoretical framework, we endeavored to make explicit how geometry – which for hundreds of years seemed to be the prototype of empirical mathematics, describing physical space – developed into the prototype of a formalistic mathematics as formulated in Hilbert’s Foundations of Geometry in 1899 (cf. Fig. 4.)

Figure 4 The historical and philosophical development of mathematics along the development of geometry.
Consequently, we helped students (or, aimed to help them) on their way to develop an understanding for different natures of mathematics, in particular, modern ones taught at the university level.

In the seminar course we began this component of instruction with Euclid’s *Elements*; they show what a deductively built piece of mathematics, describing physical space, looks like in a prototype manner. Here we prompted the students to display in a diagrammatic manner how Pythagoras’ theorem can be traced down to Euclid’s five postulates. (cf. Fig. 5, the numbers indicate the number of the proposition within Euclid’ *Elements*). This activity was selected based upon the 2013 survey results, which showed that a significant number of students were not familiar with a deductive structure after one year of university mathematics.

*Figure 5. The architecture of Pythagoras’ theorem.*

It was important for the overall goal of the seminar that the *Elements* gave reason to discuss status, meaning, and heritage of axiomatic systems. This enabled us to focus on the self-evident character of the axioms (or, postulates) describing physical space in a true manner – and to provide insights on the surrounding real space which were accepted without proof (cf. Garbe 2001: 77).

*Figure 6. Photo of “Autobahn” taken by the first author (left); Albrecht Dürer: “Man drawing a lute” (1525), (middle); photo of Albrecht Dürer Activity during seminar, taken by third author (right)*
Projective geometry was the next example on our way (in the seminar) to a modern understanding of geometry. Starting with the question of whether other geometries, besides the Euclidean one, are conceivable, projective geometry seemed to be an ideal case (cf. Ostermann & Wanner 2012: 319-344). Related to the overall goal of the course, the notion that there exists more than one geometry fostered the idea that there is more than one (“true”) mathematics. And, this in turn serves to lead us away from the quest for one unique mathematics describing physical space (cf. Davis & Hersh 1985: 322-330).

On the one hand, we wanted students to become familiar with the idea (via the Albrecht Dürer Activity, cf. Fig. 6) that projective geometry seems to be so intuitive and evident when looking at its origins in the vanishing point perspective (arts). On the other hand, projective geometry adds new objects to the Euclidean geometry (esp. the infinitely distant points on the horizon) and its place in the seminar introduced the students to the insight that all parallels may meet eventually. Additionally, with projective geometry the students encountered a further axiomatizable geometry, which also possessed particular properties that finally influenced Hilbert to ultimately design a formalistic geometry that was free of any physical references (cf. Blumenthal 1935: 402). Julius Plücker saw in the 19th century as one of the first that theorems in projective geometry hold if the terms “straight line” and “point” are interchanged. This so-called principle of duality gave a clear hint that the nature of geometrical objects may be irrelevant and that it is the relations between these objects that matter. (cf. Fig. 7)

Figure 7. Example for the principle of duality: Theorem of Pappus-Pascal: Six points (red) incident with two lines (blue) – the points (green) which are incident with opposite lines of the hexahedron are collinear (green line). Theorem of Brianchon: Six lines (red) incident with two points (blue) – the lines (green) which are incident with opposite points of the hexahedron are copunctal (green point)
The next case that students examined was a revolutionary step towards a formalistic formulation of geometry that comprised the development of the non-Euclidean geometries, and which was connected to the names Janos Bolyai (1802-1860), Nikolai Ivanovitch Lobatchevski (1792-1856), Carl Friedrich Gauß (1777-1855), or Bernhard Riemann (1826-1866) (cf. Garbe 2001, Greenberg 2004, Trudeau 1995 on their historical role regarding non-Euclidean Geometries).

In fact, the non-Euclidean geometries developed from the “theoretical question” around Euclid’s fifth postulate, the so-called parallel postulate:

Let the following be postulated: [...] That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which the angles are less than two right angles. (Heath et al. 1908)

Compared to the other postulates like the first, “to draw a straight line from any point to any point”, the fifth postulate sounds more complicated and less evident. This postulate cannot be “verified” by drawings on a sheet of paper as parallelism is a property presupposing infinitely long lines. In the words of Davis, Hersh and Marchiotto (1995: 242), “it seems to transcend the direct physical experience”. In history this was seen as a blemish in Euclid’s theory and various attempts have been undertaken to overcome this flaw. On the one hand, different individuals tried to find equivalent formulations, which are more evident (e.g. Proclus (412-485), John Playfair (1748-1819))⁵. On the other hand, several mathematicians tried to deduce the fifth postulate from the other postulates so that the disputable statement becomes a theorem (which does not need to be evident) and not a postulate (e.g. Girolamo Saccheri (1667-1733), Johann Heinrich Lambert (1728-1777)). (cf. Davis & Hersh 1985: 217-223, Greenberg 2004: 209-238, Struve & Struve 2010)

In contrast in the 18th and 19th century, Bolyai, Lobatchevski, Gauß, and Riemann experimented with negations and replacements of the fifth postulate guided by the question of whether the parallel postulate was logically dependent of the others (cf. Greenberg 2004: 239-248). If this would have been true – Euclidean geometry should actually work without it – what it does, in a sense that no inconsistencies occur.

⁵ To Proclus, who was amongst the first commentators of Euclid’s Elements in ancient Greece, already formulated doubts on the parallel postulate and formulated around 450 an equivalent formulation (cf. Wußing & Arnold 1978: 30). Playfair’s formulation (1795), “in a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point”, is quite popular today (cf. Prenowitz & Jordan 1989: 25, Gray 1989: 34).
Figure 8. Visualizations regarding different geometries. Elliptic, Euclidean and Hyperbolic Geometry. (naiadseye, 2014)

But this logical act leads to conclusions that differ from those in Euclidean geometry. For example:

- In hyperbolic geometry the sum of interior angles in a triangle sums to less than 180°, in elliptical geometry to more than 180° (cf. Fig. 8)
- The ratio of circumference and diameter of a circle in hyperbolic geometry is bigger than π, in elliptical geometry smaller than π.
- In hyperbolic as in elliptical geometry triangles which are just similar but not congruent do not exist.
- In hyperbolic geometry there is more than one parallel line through a point P to a given line g and in elliptical geometry there are no parallel lines at all. (cf. Davis & Hersh 1985: 222, Garbe 2001: 59)

Working with texts and sources regarding the process of discovery of the non-Euclidean geometries had an important impact on students’ belief system. The 2013 survey results indicated that the so-called “Euclidean Myth” (Davis & Hersh 1985) was widely prevalent: to many first-year university students mathematics is a monolithic block of eternal truth; a theorem, once proven, necessarily holds in every context. With the discovery of the non-Euclidean geometries, it became apparent in history that there was no such truth in an ontological sense. In contrast, there seems to be multiple such truths, depending on the context in which you work. We used a discussion of Gauß’s qualms to publish his results on non-Euclidean geometry, afraid of being accused of doing something suspect, or the (probably legendary) story (cf. Garbe 2001: 81-85) that he tried to measure on a large scale whether the world is Euclidean to help the students become amenable to the revolutionary character of his discoveries. Following Freudenthal’s (1991) idea of guided reinvention, recapitulating the
history of humankind seems to bear quite fruitful perspectives for the development of individual belief systems.

Finally, from the discussion of the non-Euclidean geometries students investigated questions which led to Hilbert’s formalistic turn. If there was more than one consistent geometry, which one was the true one? This question is closely linked to the question, what is mathematics?

3) Exploration of Hilbert’s approach. (Or, what characterizes modern formalistic mathematics?)

Hilbert actually gave an answer to this problem – not only in a philosophical and programmatic way but also by formulating a geometry “exempla trahunt” (Freudenthal 1961: 24), a discipline that was seen for ages as the natural description of physical space, in a formalistic sense and characterized by an axiomatic structure. The established axioms are fully detached and independent from the empirical world, which leads to an absolute notion of truth: mathematical certainty in the sense of consistency. Thus, with Hilbert the bond of geometry to reality is cut. This came to life in the seminar when students read Hilbert’s Foundations of Geometry (1902, see Fig. 9) in detail.

Figure 9. The famous first paragraph of Hilbert’s (1902) Foundations of Geometry.

Hilbert did not give his concepts an explicit semantic meaning; he spoke independently from any empirical meaning of “distinct systems of things”. Consequently, intuitive relations like in between or congruent do not have an empirical meaning but are relations fulfilling certain formal properties only (cf. for example, Hilbert & Bernays 1968: §1, Greenberg 2004: 103-129).

As we all know, the discussion of nature of mathematics did not come to an end with Hilbert. Thus, the course ended with discussions of texts taken from What is Mathematics, Really? (Hersh 1997). Hersh understood “mathematics as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context” (xi), which created a balanced view.

However, nobody will deny that formalism in Hilbert’s open-minded version had a lasting effect on the development of mathematics. As a consequence, today’s university mathematics has the freedom to be developed
without being ‘true’ in an absolute sense anymore (cf. Freudenthal 1961), but nevertheless including the possibility to interpret it physically again.

In the meantime, while the creative power of pure reason is at work, the outer world again comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus at the same time advance most successfully the old theories. And it seems to me that the numerous and surprising analogies and that apparently prearranged harmony which the mathematician so often perceives in the questions, methods and ideas of the various branches of his science, have their origin in this ever-recurring interplay between thought and experience. (Hilbert 1900: English translation in Reid 1996: 77)

It is this openness and freedom of questions of absolute truth, which Hilbert replaced by the concept of logical consistency that made mathematics so successful in the 20th century (cf. Freudenthal 1961: 24, Garbe 2001: 100-109, Tapp 2013: 142).

This makes again quite clear that modern mathematics after Hilbert is on epistemological grounds, completely different than (historical) empirical mathematics and of course, mathematics taught in school. Whether the first is grounded on set axioms and the notion of mathematical certainty (inconsistency), the second and third are grounded in evident axioms – thus describing physical space including a notion of (empirical) truth, resting essentially on induction from experience.

4) Summary discussion and reflection

The final session of the seminar entailed a whole-group discussion in which we sought to connect insights gained from the historical perspectives with the individual participants’ mathematical biographies. We first reminded students about the preliminary discussions regarding different personal belief systems that occurred in the first session of the seminar. The intention was that the transparency on the historical problems that led to a modern abstract understanding of mathematics can therefore lead to an understanding of what happens if students live through this revolution on epistemological grounds as individuals, thus opening differentiated views on the transition problem.

As an example, the first author – while leading the concluding discussion of the seminar course – prompted students with:

You have described, that [it] is all abstract; there is no application. […] You have also said, that it is somehow not too bad, because it is also important...If you are watching the reality, that’s what I want to remind you, at the differences between high school-university, that you also described again … (fourth author’s translation)

To this, one student shared:

If first-year students go to university, then they have a completely different concept map in their mind and for example, [they] have the understanding that a
graph is always a function. But in fact that is not right. And if the lecturer talks about such definitions, [students] will say: “Huh? I have never ever seen something like this in my whole life,” and principally they know, they can connect it to their knowledge, but they need simply someone who explains to them ok, the function is not the graph. Also they need principally a dictionary for high school to university, where you can look up the concepts. (fourth author’s translation)

In this student’s personal mathematical biography, then, there was a clear gap between the mathematics experienced in high school when compared to that at university. And, the gap was so pronounced that a sort of translation device – “a dictionary for high school to university” – was required to make sense of the different concepts.

6. SUMMARY

Although the primary intent of this article was to share the usefulness of the intensive seminar we conceptualized and implemented with one group of university mathematics students at a German university in spring 2015, another aim was to share initial reflections on the data we gathered to determine whether an intervention longer than a three-day seminar was both warranted and necessary.

The group of students who participated in the seminar was heterogeneous with respect to age and semesters at university (see Table 1), which gave us multi-perspective views on the success of the seminar and a deeper insight of the transition problem. Numerous data sources will inform the construction of six case studies which will describe the ‘state of the transition’ that the participants experienced – and are still experiencing – with respect to the transition from school to university mathematics. The data sources include pre- and post-surveys (measures of beliefs and perceptions of mathematics, content items, and demographic information), video and audio recording of the three-day seminar, essays submitted by all seminar participants, audio recording of interviews of six seminar participants, observation notes (third author), and various seminar artifacts (e.g. daily debrief notes completed with students, response cards to open, anonymous prompts).

However, as with the preceding sample revelations, further evidence – in the words of the students – revealed that they could articulate the transition problem adequately and that they desire a solution as they contemplated the next “abstraction shock” they will encounter. For example, in the summary discussion, one young woman declared:

Even the problem with [limits], that was also described... and now it appears for me, as if the Anschaulichkeit and the applications are the

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6 As we have stated throughout, we intended for this article to present the theoretical foundation for and a description of the ÜberPro seminar that we implemented in February 2015. We have purposefully reserved the presentation of signature cases of student participation in the seminar for subsequent publications.
reasons; I mean the application in school, [is] the reason, that we now have problems while the transition process to university. And then I don’t understand, why this is in all the books of didactics nowadays, that using applications is very good and that instead, it is the problem for the transition to university.

Another student observed in the essay assigned at the conclusion of the seminar that:

All in all, the transition problem in mathematics is quite rightly an often-discussed topic, which seems is hard to solve. For many students the transition from school to university is [difficult] because of the following aspects: the changes in teaching and learning, the change in the character and beliefs on mathematics ((naive) empirical to deductive-mathematical), the pressure to perform and the [subsequent] loss of motivation. That’s why they fall into a nearly never-ending ravine, from which they have to find a way out, for overcoming the transition successfully. If they fail at this, they break up their studies. For me the transition from school to university was and is also not very easy.

Still another student shared in his/her essay response that:

Everything we discussed in [the] seminar led me to believe that it is crucial to understand the transition problem with the help of mathematical history. I would have liked to have some more practical advances in how to use this situation later as a teacher. (I know that [was not] the aim of this class and the research is probably at the very beginning but at some times we could have spent the time in a better way.)

Thus, it was clear to us that there is much more work that we can do in responding to the seminar course students’ needs. Indeed, mathematical history can provide support in negotiating the second gap that university mathematics students encounter when they transition to teaching mathematics. One such support is to provide concrete ways in which mathematics teachers can draw upon particular moments in the historical development of a collection of related mathematical ideas (as in the case of geometry in the ÜberPro seminar). However, another support includes the way in which history of mathematics contributes to a teacher’s mathematical knowledge for teaching, particularly contributions to horizon content knowledge (Clark 2012).

6.1 Implications for next steps

For school purposes – from a well-informed mathematics educator’s point of view – nothing speaks against doing mathematics in an empirical way. Indeed, history has shown that empirical mathematics was a decent way to develop mathematical knowledge and the experimental natural sciences generate knowledge comparably. Yet approaches to bring formalistic mathematics into school classrooms have failed miserably (cf. the New Math Initiative, Why
Johnny Can’t Add (Kline 1974)). Moreover, we cannot step away from teaching mathematics in a theoretical way at universities. In contrast, the intensive seminar course that we implemented sought to make tangible, understandable, and explicit to first-year university students that the transition from school mathematics to university mathematics is an epistemological obstacle.

Hefendehl-Hebeker (2013: 80) found quite comparably:

[…] a principle difference between school and university is that at university with the axiomatic method a new level of theory formation has to be reached, and thus it follows that the discontinuity cannot be avoided.

So if the discontinuity cannot be avoided, what can teachers and students at university gain from a seminar course like the one described here? We found that significant potential lies in the following areas:

1. The historical excursions do not only focus on the beliefs aspect but also demonstrate and involve critical mathematical activities, especially regarding deductive reasoning within the frameworks of consistent mathematical theories.

2. Teachers and students should become aware of the extent of the transition problem, and that the problem’s solution is not as easy as repeating particular secondary school mathematics, as many approaches (and deficit models) seem to suggest. Instead, a revolutionary act of conceptual change is required and this work does not occur overnight and needs guidance. The historical questions that led to the modern understanding of mathematics are too sophisticated and waiting for students to develop these for themselves is a particular burden on top of all the other factors of beginning mathematical study at university. The approach of initiating these questions explicitly within the framework we described here may support a more adequate and prompt change of belief system, which in turn holds promise for addressed both forms of “abstraction shock” experienced by secondary mathematics teachers.

3. The seminar course has the power to sensitize for critical communication problems. Teachers and students should acknowledge that when talking about mathematics, using the same terms might not imply talking about the same things. For example, students may come to university from school having learned calculus in an empirical context such that functions might be equivalent to curves. This might imply that properties like continuity or differentiability are empirical and can be read from the sketched graph of the function (comparable to 17th century mathematicians). The university lecturer, on the other hand, probably has a general abstract notion of function implying a completely different notion of mathematical reasoning and truth. In particular, lecturers should repeatedly check if the knowledge of their students is still bound to (single) objects of reference. The same holds for the students eventually leaving university and starting as secondary school
mathematics teachers: they should be aware that what they consider from an abstract point of view their students may instead possess visualizations of abstract notions as the reference objects.

NOTES

1. Anschauung: The meaning of the prominent German term Anschauung has two different connotations. It can mean something close to ‘empirical perception’ or something like an ‘inner mental image’ (according to Immanuel Kant). Heuser (2009) referred to the aspect of empirical perception.

2. Empirical: The authors use the word empirical in the sense as it is used in the concept of “empirical theories” in philosophy of science, which is close to natural scientific theories. That means that some concepts of a theory have real/physical/empirical reference objects and the propositions of the theory can be checked by experiments in reality (Hempel 1945, Stegmüller 1987).

3. Formalistic: The authors use the word formalistic in the Hilbertian sense. That means a (mathematical) theory is formalistic if all primitive concepts of the theory are (logical) variables and the axioms of the theory are not sentences but sentential functions with the primitive concepts as variables arguments (cp. C. G. Hempel 1945). By virtue of a physical interpretation of the originally uninterpreted primitives empirical models of the formalistic theory are defined. This is the relation between a formalistic mathematics and empirical science.

Figure Acknowledgements:

Fig. 1. Data from a survey made by Ingo Witzke in 2013.
Fig. 2. Graphical derivative. Graphic from Griesel, H. et al. (Hrsg.): Elemente der Mathematik (EDM), Einführungsphase – Braunschweig: Schroedel 2010: 203.
Fig. 3. Three excerpts of different textbooks for comparison. University course textbook “Königsberger 2001: 34” (top left), school textbook “Lambacher Schweizer 2009: 55” (bottom left), historical text “Leibniz, Acta Eruditorum”, 1693 (right).
Fig. 4. Historical development as one basis of the seminar. Created by Ingo Witzke and Gero Stoffels.
Fig. 5. The Architecture of Pythagoras theorem. Graphic by S. Schlicht (University of Cologne) 2014.
Fig. 6. Photo of “Autobahn” taken by Ingo Witzke (top); Albrecht Dürer: “Man drawing a lute” (1525), (bottom left); photo of Albrecht Dürer Activity during seminar, taken by Kathleen Clark (bottom right).
Fig. 7. Example for the principle of duality: Theorem of Pappus-Pascal: Six points (red) incident with two lines (blue) – the points (green) which are incident with opposite lines of the hexahedron are collinear (green line).
Theorem of Brianchon: Six lines (red) incident with two points (blue) – the lines (green) which are incident with opposite points of the hexahedron are copunctal (green point). Graphic created by Horst Struve and Ingo Witzke.

Fig. 8. Angle sums in different geometries: Internet source retrieved November 1, 2015 from the World Wide Web: https://naiadseye.files.wordpress.com/2014/10/euclidean-understand-e1414490051530.png?w=470&h=289 (Stand, 2015).

Fig. 9. First paragraph of Hilbert’s *Foundations of Geometry* (Hilbert 1902).

REFERENCES


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