PRIMARY SOURCES AND HISTORY-BASED PROBLEMS ABOUT ISOPERIMETRY: A USE OF MATHEMATICS HISTORY IN GRADE SIX[1]

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ABSTRACT

In this paper, we report on the use of one historical note and two primary sources, an extract from Pappus' *Collection* and an extract from Polybius' *Histories*, in the context of an instructional intervention focused on isoperimetric figures and area-perimeter relationships. The participants were 22 sixth graders, aged 11-12. The research findings we present here are based on classroom observations, on the worksheets used during the intervention and on personal interviews with the students. During the intervention, the students solved problems, which were based on the sources. Twenty-one of the 22 students considered the problem which was based on Pappus' text to be more interesting than the problems that they were usually asked to solve in mathematics. In addition, the students' ratings of the texts indicate that the extract from Pappus was the text that they liked most. We also examine the various ways through which the particular use of mathematics history affected the development of the students' personal Geometrical Working Spaces.
Keywords: History of mathematics, Primary sources, Isoperimetric figures, Area, Geometrical Working Space

1. INTRODUCTION

This paper presents some findings from a larger research study linking the use of historical sources in mathematics education with the Geometrical Working Spaces theoretical framework (Kuzniak 2006), in the context of an instructional intervention focused on isoperimetric figures and area-perimeter relationships. In the paper, we focus on how the sources were used and we discuss the students' views both on their learning and on the use of the particular historical sources, and the various ways through which the particular use of mathematics history affected the development of the students' personal Geometrical Working Spaces.

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1.1. History of mathematics and mathematics education

Regarding the use of mathematics history, on the one hand, there are theoretical objections and practical difficulties. For example, it has been argued that students often dislike history and that the history of mathematics could confuse students (Jankvist 2009, Tzanakis et al. 2000). Practical difficulties include the lack of teaching time and material and the teachers' lack of expertise. Moreover, the use of mathematics history could not be easily assessed, so it would not attract students' attention.

On the other hand, it has been argued that mathematics history can motivate students and contribute to the teaching of specific mathematical content (Jankvist 2009, Tzanakis et al. 2000). Moreover, learning about the difficulties, errors and misconceptions that arose in the history of mathematics could be beneficial to students in terms of emotions, beliefs and attitudes; on the other hand, this kind of knowledge helps teachers to anticipate students' possible difficulties and to develop or adapt history-based problems and other instructional material that could help students overcome these difficulties. Also, mathematics history shows the role of individuals and the role of different cultures in the evolution of mathematics and indicates that mathematical concepts were developed as tools for organizing the world. Finally, mathematics history enables the connection between mathematics and other subjects.

Concerning the relationship between students' difficulties and the difficulties encountered in mathematics history, there are different approaches. Through the concept of epistemological obstacle, Brousseau (2002) emphasized the role of a piece of prior knowledge, which, depending on its structure, has particular advantages but also leads to particular errors. Contrarily, Furinghetti and Radford (2008) emphasized the role of culture and argued that school prepares the unpacking of a tradition established over centuries. Finally, according to the conceptual change framework, children's initial theories can emerge through the children's interaction with the physical environment and with the cultural tools (Vosniadou & Vamvakoussi 2006). Thus, similarities between children's difficulties and the difficulties encountered in history could possibly be related to the use of similar cultural tools or to children's perception of the environment; this seems to be particularly interesting in the case of elementary geometry, considered as the science of space (Kuzniak 2006).

As regards the ways of using mathematics history, the most common way is the use of historical notes, i.e. texts that are written for teaching purposes and may include names, dates, biographies, anecdotes and stories (Jankvist 2009; Tzanakis et al. 2000). Worksheets, historical problems, and primary and secondary sources are also forms of using history. The various history uses can also be combined for designing teaching and learning sequences (packages) and projects, which may be short or more extensive and more or less relevant to the curriculum.

The use of primary sources is both demanding and time-consuming, and it is often difficult to assess the results (Jahnke et al. 2000). The teacher may need to translate or modify the text, but such adaptations should not deviate far from the original text. A primary source may be introduced directly (without prior preparation) or indirectly, e.g. after problem solving. In short, there is not only one teaching strategy for the use of primary sources; therefore, the most appropriate strategy should be chosen.

1.2. Area-perimeter relationships in ancient Greek mathematics

There is sufficient evidence to suggest that area-perimeter relationships have caused difficulties in the past. For example, Polybius from Megalopolis (2nd c. BC), in the ninth book of his treatise *Histories*, argued that army generals should have knowledge of astronomy and geometry, and to support his claim, he wrote: "Most people infer the size of the aforementioned [cities and camps] only from the perimeter. (....) The reason of this is that we do not remember the geometry lessons we were taught in our childhood" (*Hist.* 9.26a.1-4, Büttner-Wobst ed.).[2] Furthermore, he gave two examples: the first concerns the comparison between Sparta and Megalopolis, while the second concerns a hypothetical town or camp which has a perimeter of 40 stadia but is twice as large as another with a perimeter of 100 stadia.

According to Walbank (1967), 'the size' is the area of each city. Moreover, the first example is of particular historical interest, since the comparison seems not to be confirmed in the case of area, at least with the existing archaeological findings. On the contrary, the second example refers to an extreme case and is mostly of mathematical interest. In any case, Polybius' reference to geometry is a characteristic example of the way that ancient writers used mathematics to present their accounts as superior in terms of accuracy and reliability (Cuomo 2001).

Polybius' reference to 'geometry lessons' shows that area-perimeter relationships had already been an object of study for mathematicians. In the *Elements*, Euclid had already proved that parallelograms on the same base or on equal bases, and between the same parallels are equal to one another and then he proved the same for triangles (I.35-38). These theorems imply that the length of the contour of a parallelogram or triangle does not determine the extent of its surface; this is why, according to Proclus, these theorems caused astonishment to non-experts (Heath 1921).

Isoperimetry was also the object of Zenodorus' work (probably 2nd c. BC). His treatise on isoperimetric figures has not survived; however, on the basis of what Theon wrote later, Zenodorus proved that of all regular polygons with equal perimeter, the largest is the one having the greatest number of angles, and that if a circle and a regular polygon have equal perimeter, then the circle is larger (Cooke 2005, Heath 1921). Furthermore, he showed that of all

[2] Book 9 survives in fragments, and there have been different views concerning the order of the fragments. In other editions or translations, this passage is part of 9.21. In Büttner-Wobst's edition it is a part of 9.26a, and Walbank (1967) considered this order to be more coherent.

isoperimetric polygons with the same number of angles, the largest is the equilateral and equiangular, but he partially based his proof on a lemma that had not been proved in a general way.

Isoperimetry is also the topic of Book V of Pappus' *Mathematical Collection* (4th c. AD). The first part of the book concerns plane figures and begins with an introduction, which is characterized of literary merit (Cooke 2005, Heath 1921) and stimulates the interest of the reader; its topic is the hexagonal shape of the cells of honeycombs. Pappus' explanation of the shape is teleological, as he claimed that bees choose this shape on purpose. At the end of the introduction, Pappus formulated a mathematical problem:

> *Bees then know only what is useful to them. That is, that the hexagon is greater than the square and the triangle, and can hold more honey, for the same expenditure of material for the construction of each one. We, however, claiming to have a greater share of wisdom than bees, will investigate something greater. That is, that of all equilateral and equiangular plane figures having equal perimeter, the one which has the greatest number of angles is always greater. And the greatest of all is the circle, whenever it has perimeter equal to them. (Mathematical Collection V.3,* Hultsch ed.*)*

According to Cuomo (2000), Book V was probably situated in the context of rivalries for the appropriation of tradition, for the acquisition of reputation and for the gaining of new pupils. Bees were frequently used as an example by philosophers too; for Pappus, the difference between bees and humans is that bees have limited, useful and intuitive knowledge, whereas humans are both capable of and interested in proving. Thus, the introduction points out to the need for proving the isoperimetric theorems. The proof process, which follows, is situated in the context of the Euclidean tradition. Furthermore, although there is no reference to Zenodorus, it seems that Pappus followed Zenodorus' work, especially in the case of plane figures, but also added his own propositions and proofs (Heath 1921).

Pappus' introduction about bees is also related to the problem which was later known as the honeycomb conjecture. According to the conjecture, which has been proved more thoroughly by Hales (2001), "any partition of the plane into regions of equal area has perimeter at least that of the regular hexagonal honeycomb tiling" (p. 1).

1.3. Theoretical framework for designing the intervention

Work with isoperimetric figures, that is geometric figures with equal perimeters, involves the concepts of perimeter and area and their relation. Regarding these concepts, prior research (Douady & Perrin-Glorian 1989, Moreira-Baltar & Comiti 1994, Vighi 2010, Woodward & Byrd 1983, Zacharos 2006) has shown that students often use formulas at the expense of other strategies, make errors when applying them and do not understand the result, and confuse area and perimeter; they also believe that a smaller/equal/greater perimeter implies a smaller/equal/greater area respectively, and vice versa, and this misconception often reappears after instruction. Furthermore, the difficulties related to area-perimeter relationships are observed even in secondary school students and adults (Kellogg 2010, Woodward & Byrd 1983).

Regarding the concept of area, it has been argued that students need to understand that area is an attribute (Van de Walle & Lovin 2006). The following strategies have been recommended: a) area measurement with the use of two-dimensional units, b) comparison of areas of different figures, c) superposition of a surface onto another and reconfiguration of one of the surfaces, and d) examination of area-perimeter relationships (Douady & Perrin-Glorian 1989, Nunes, Light, & Mason 1993, Van de Walle & Lovin 2006, Zacharos 2006). It is also worth noting that in the USA the examination of area-perimeter relationships is recommended for Grade 3 or above (Common Core State Standards Initiative 2010, Georgia Department of Education 2014, North Carolina Department of Public Instruction 2012, Van de Walle & Lovin 2006).

In this research study the concepts of area and perimeter were examined from the standpoint of geometry, so we used the Geometric Working Spaces theoretical framework (Kuzniak 2006, 2015). A Geometric Working Space (GWS) is a space organized in a way that makes it possible for the user of the space (mathematician or student) to solve a geometric problem. Therefore, problems are the reason of existence of GWSs. The framework distinguishes three levels: a) the reference GWS, which is determined by a particular community of mathematicians or, in education, by the curriculum, b) the appropriate GWS, which is designed by a teacher for a particular class, and c) the personal GWS, which is developed by the final user, in our case each student. In addition, there are three paradigms. Here, we are mainly interested in Geometry I (GI), wherein experimentation is dominant, and practical proofs, measurement, the use of numbers and approximate answers are allowed, and in Geometry II (GII), whose archetype is the classical Euclidean geometry.

The GWS's epistemological plane includes three components: a) a real space with its geometric objects, b) a set of artifacts, and c) a theoretical frame of reference with the definitions and the properties of the objects (Kuzniak 2015). A second and cognitive plane includes three kinds of processes: visualization, construction and proof. Visualization includes the reconfiguration of figures, which may be performed materially or with the use of reorganizing lines or only by looking (Duval 2005).

Concerning students' misconceptions, Brousseau (2002) has argued that overcoming an obstacle requires the involvement of students in solving selected problems, through which they will realize the ineffectiveness of a piece of knowledge or conception. He noted, however, that problems should be chosen in a way that students are motivated and, subsequently, act, discuss and think so as to solve them. Another strategy which can help students change their ideas is the use of refutation texts (Tippett 2010), i.e. texts which refer to a prevalent alternative idea, stressing that it is incorrect. For the use of these texts, a combination with discussions and other activities is recommended, because changing ideas is difficult and no text alone is sufficient to achieve this with all students. Finally, it is recommended that teaching should close with a metacognitive phase, during which "the teacher asks the students to describe their old and their new knowledge and to realize its differences" (Kariotoglou, 2006: 36).

We referred above to the role of students' motivation in solving problems and we noted that motivation is a usual goal when using the history of mathematics. Besides, from the viewpoint of motivational psychology, Pintrich and Schunk (2002) have recommended, among others, the use of original source material. Regarding the features of the texts that stimulate interest, Schraw, Bruning and Svoboda (1995) highlighted the role of vividness and of ease of comprehension. Moreover, prior research has found that students and teachers argued that when a text is read aloud by the teacher, it becomes more interesting, and comprehension becomes easier (Ariail & Albright 2006, Ivey & Broaddus 2001). Other factors that could stimulate interest are: novelty, group work, hands-on activities, some themes related to nature, meaningfulness and the balance between the degree of challenge and the level of knowledge and skill of a person (Bergin 1999, Mitchell 1993, Pintrich & Schunk 2002).

2. METHOD

As already mentioned, this paper presents some findings from a larger research study. In the paper, we focus on two questions:

1. In what ways was the particular use of mathematics history related to the development of the students' personal Geometrical Working Spaces?
2. What were the students' views both on the particular use of mathematics history and on their learning?

The research was conducted in Thessaloniki, Greece, and the participants were 22 sixth graders, aged 11-12. The findings we present here are based on classroom observations, worksheets and personal interviews with the students. The instructional intervention was implemented by the first researcher in the regular classroom of the students. An exception was the class period allotted to the solution of the main mathematical problem, for which we decided not to have the groups of students work simultaneously in the regular classroom but to have each group work for one class period in another classroom of the school. This was decided in order to enable the observation of the students' work and of the difficulties they faced. Thus, the whole intervention consisted of six class periods in the regular classroom and one class period for each group in another classroom.

The whole research project also included personal interviews with the students before and after the intervention. In this paper, we focus on the interviews conducted after the intervention and, in particular, on the questions asking the students to provide some further explanation concerning their views on the particular use of mathematics history.

2.1. Design of the appropriate GWS

The intervention was implemented prior to the teaching of the geometry unit, because the textbook's emphasis on formulas and the learning of area formulas for general triangles and trapezoids would affect the students' personal GWSs, favouring the use of formulas at the expense of other strategies.

Concerning mathematics history, we selected two primary sources: the first was the introduction to the first part of Book V of Pappus' *Mathematical Collection* (V.1-3) and the second was an extract from Polybius' *Histories* (9.26a.1-6). In addition, we decided to use a historical note entitled *Geometry* and included in the sixth grade textbook (Kassoti, Kliapis, & Oikonomou 2006: 136).

Since primary school students do not know ancient Greek, the sources were presented in translation. During the translation, we used words and phrases as close as possible to the original texts, while in some cases we used shorter sentences, so that the translated texts were both suitable for the students and close to the original (Jahnke et al. 2000). Furthermore, on the basis of the objectives of the intervention, we did not include in the extract from Pappus the vocative address "most excellent Megethion" (*Mathematical Collection* V.1), the closing of the introduction including the reference to the circle (V.3), and the detailed verbal proof of the fact that only three regular figures can completely cover a surface without gaps or overlaps (V.2).

The extract from Pappus, as a historical source, was not introduced directly, but after the use of the historical note. More specifically, work with the historical note included reading it, discussing briefly about the origin and development of geometry and providing additional information about Pappus' life and his historical period. As regards Pappus' text, a different approach was selected: formulation of questions by the teacher, followed by a teacher read-aloud of the text, and discussion based on the initial questions. Then, the teaching plan included providing the students with a copy of the extract and asking them to underline words and phrases related to mathematics. The goal was to provide or help the students activate the definitions and geometric properties needed for developing their GWSs, namely definitions of polygon, regular polygon, equilateral triangle, square and regular hexagon, and definitions of perimeter and area; also which regular figures completely cover a surface without gaps or overlaps, which figures are called isoperimetric and how demonstration is related to mathematics.

Work with the text was followed by the formulation of a geometric problem asking the students to examine if Pappus was right in stating that a cell having the shape of a regular hexagon holds more honey than other figures suitable for tessellation. The students were asked to solve the problem in groups and with different methods:

1. Direct area comparison: superposition of a surface onto another and reconfiguration of one of the surfaces.
2. Indirect area comparison: tiling of equal surfaces (inverse proportion: the

shape which is used fewer times for the tiling of equal surfaces is greater).

3. Area measurement with the use of a transparent grid (square-counting).
4. Area calculation with the use of formulas.

The objects of the real space were regular polygons with three, four and six angles, and irregular polygons (elongated rectangles), in a material form (cardboard), in order to facilitate the reconfiguration of the shapes. Since the length of the sides was not given, the students needed to measure the sides, so as to calculate the perimeter of each shape, and then they were asked to apply the proposed method of area comparison (GI). The tools selected to be available (where appropriate, depending on the method) were: triangle ruler, scissors, glue, adhesive tape, transparency film with a square grid printed on it, marker pen, pencil, rubber eraser and calculator. In addition, we prepared one worksheet for each group, aiming, firstly, to provide through a set of questions particular steps for solving the problem and, secondly, to help students present their findings in the classroom.

The institutionalization of the new properties was followed by the use of the extract from Polybius. Work with the second source included a discussion about area-perimeter relationships, and two other activities. The first one asked what the shape and the dimensions of two cities could be, if the one had a perimeter of 40 stadia but twice the area of the other having a perimeter of 100 stadia. The second activity was called 'Neighborhoods of Thessaloniki' and involved eight isoperimetric figures, which represented neighborhoods (Appendix, Fig. 1).

More specifically, each pair of students was given two figures, which were printed on a sheet of paper, along with the length of each side in metres. The students were asked to calculate the perimeter of each figure and to deduce, without calculation, if an area was smaller than, equal to, or larger than the other and why.

Regarding perimeter, the students needed only to add the given lengths and realize that the figures were isoperimetric. Regarding area, they had to develop the theoretical pole of their GWS (GII), applying the institutionalized conclusions which were based on the honeycomb problem. Then, each pair of students was asked to present their answer and check its correctness by performing measurements (GI) via a computer connected to a projector and with the use of a Geogebra applet designed for the activity. In the applet, a map of Thessaloniki was inserted as a background and the eight figures were on the same scale as the map. Finally, the students were asked to put all the figures on a board from the smallest to the largest in area, noting that eight different figures had equal perimeter but different area, that the largest in area was the regular figure having the greatest number of angles, and that the smallest was the most elongated figure.

3. RESULTS

3.1. Implementation of the appropriate GWS and students' difficulties

Regarding the extract from Pappus, we noticed two interrelated behaviours. First, when students were asked to find in the text words and phrases related to mathematics, none of them mentioned the word 'proof'. Secondly, after working with the text, several students seemed convinced that Pappus was right and they agreed a priori that the hexagon will be larger.

In the honeycomb problem, all groups correctly arranged the figures in increasing order of area. The main difficulties they faced while solving the problem were the following:

- Superposition-reconfiguration: the relatively most difficult comparison was between the hexagon and the square (Appendix, Fig. 2). Overall, however, this method was the easiest.
- Tiling of equal surfaces: the students understood the rationale of the method when the teacher provided the hypothetical example of two identical rooms with different tiles. When counting the number of shapes used, we noticed more difficulties in the case of the hexagon, since there were parts that were smaller or greater than half the hexagon (Appendix, Fig. 3), and the students had to recompose these parts by looking (Duval 2005).
- Square-counting: at first, the students did not remember that in previous grades, to find the area of a figure, they counted squares, in grids which were either pre-drawn on the pages of the textbooks or designed by the students. In addition, they had to find an operational way to use the transparent grid, which was new to them as a tool. The most difficult point was the counting of small squares in the case of the hexagon (Appendix, Fig. 4); an advanced solution was given later and involved the reconfiguration of the entire hexagon, in a way that two rectangles were formed.
- Calculation: certain shapes needed to be reconfigured so as to form shapes whose area could be calculated with the already taught formulas (Appendix, Fig. 5). There were difficulties regarding the choice of the appropriate formula, the reconfiguration of the hexagon, and the calculation which was required when a shape had been reconfigured not with the use of scissors, but via folding.

Furthermore, some students from different groups showed area-perimeter confusion. Another obstacle for the students was the usual didactic contract, since in Greek upper elementary education hands-on activities with figures presented in a material form are, in practical terms, almost absent. Thus, some students felt the need to ask for permission to use the scissors and to fold or cut the shapes, although they had been told that they could work as they wanted, using whatever tool available they wanted.

Concerning the extract from Polybius, in the discussion which followed, the students concluded, "that a region can have greater perimeter but smaller area [as compared with another region], or smaller perimeter but greater area" (student Q) and that "if a region is greater, it's not perimeter that matters, it is area that matters" (student Z). As regards the activity 'Neighborhoods of Thessaloniki', an indicative example is the response of the students who compared the rectangle with the square: "The square is greater, because they have equal perimeter and those [figures] that are regular are greater". Likewise, in comparing the regular pentagon with the equilateral triangle, the following answer was given: "They are regular and isoperimetric the one to the other. Although they have the same perimeter, the pentagon has more angles than the triangle, thus we assumed that the pentagon is greater". On the other hand, there was a student who calculated the perimeters incorrectly and another student who was initially willing to work within GI, by reconfiguring the figures and calculating with formulas, but this was difficult, since the figures were in fact scaled representations.

3.2. Students' self-references regarding their learning

In the last worksheet used during the intervention there were several questions aiming to help the students reflect on their learning. These were not answered by all students, and there were also some non-specific answers. The rest of the answers referred:

- To area-perimeter relationships. For example, student Y wrote that an idea which she changed was that "those figures which have the same perimeter always have the same area too", while her new idea was that "area and perimeter are not related". Also, student D wrote that something which surprised him was that "small and large figures have the same perimeter".
- To ideas or processes associated with experimentation. For example, student I wrote that an idea which she changed was that "to find perimeter I believed that I should do side · side", but "I discovered that we do side + side + side + side...". Furthermore, student H wrote that something he learnt is "that I can find which figure has the biggest area without calculating it", thus showing the dominance of calculation with formulas in the students' past experiences. Likewise, student X reported that something which surprised him is "that there are so many different methods to measure which figure is bigger".
- To bees and to the shape of the honeycomb cells, as something that caused surprise.
- To the students' attitude towards geometry. In particular, student Z said that, previously, she did not love geometry, whereas after these lessons she liked it somewhat more, because she understood them. Similarly, student Y wrote that something that surprised her is that "I believed that geometry was difficult, confusing and incomprehensible, but after these lessons I found that it is easier".

In addition, the students were asked to write something they found difficult. Two students mentioned the honeycomb problem, one student mentioned the method of tiling and two others mentioned the method of square-counting, four students wrote that they found it difficult to find the area of the hexagon or of the triangle, and one student referred to the fact that "There are figures with equal perimeter". Finally, several students wrote that they did not find anything difficult or they did not write anything.

3.3. Students' assessment of the sources and of the problem

In the same worksheet, the students were also asked how much they liked each of the texts used in the lessons. The students could rate each text on a 5-point scale ranging from 1 (the least) to 5 (the maximum). Regarding the historical note, the mean score was 3.45 (SD = .91, N = 22), whereas in the case of the extract from Pappus the mean score was 4.36 (SD = .66, N = 22). Finally, regarding the extract from Polybius, the mean score was 3.85 (SD = 1.31, N = 20); we note that two students were asked to rate only the two first texts, since they had been absent from school when the extract from Polybius had been taught.

To determine whether there is a statistically significant difference as to how much the students liked the three texts, we excluded the two students who did not rate the third text (N = 20), and we used the Friedman test, which showed that the difference was significant ($\chi^2 = 6.818$, df = 2, $p = .033 < .05$). As a post-hoc test, we used the Wilcoxon Signed Ranks Test with Bonferroni correction (Corder & Foreman 2014), which showed that the difference was statistically significant in the comparison between the extract from Pappus and the historical note ($Z = -2.857$, $p = .004 < .017$), but not between the extract from Pappus and the extract from Polybius ($Z = -1.543$, $p = .123$) nor between the extract from Polybius and the historical note ($Z = -.997$, $p = .319$). We note that both the Friedman and the Wilcoxon test are non-parametric, but they are more appropriate for ratings and for small sample sizes ($N<30$) (see also: Corder & Foreman 2014).

Finally, taking into account the answers of all the students (N = 22) as regards the extract from Pappus and the historical note, the difference was again statistically significant ($Z = -3.137$, $p = .002$). Moreover, only two of the 22 students liked the historical note more, whereas 15 students liked the extract from Pappus more, and five students liked both texts equally.

In the same worksheet, the students also reflected on the difficulty and the interest caused by the honeycomb problem, as compared with the problems that they were usually asked to solve in mathematics. In this question, 11 out of the 22 students considered the honeycomb problem to be easier than usual, whereas eight said that it was more difficult, and three answered that it had the same degree of difficulty. However, 21 of the 22 students found this problem more interesting than the usual problems, and one student answered that it was equally interesting.

A comparison of the self-reported degree of the problem's difficulty with the method used by the students in solving it, shows that none of those who performed superposition-reconfiguration considered the problem to be more difficult. In contrast, three quarters of those who used square-counting considered the problem to be more difficult. Furthermore, a recoding of the responses (1: more difficult; 0: same degree of difficulty; -1: easier), shows that, on average, the students who performed superposition-reconfiguration or tiled equal surfaces considered the problem to be easier (average degree of difficulty -.5 and -.3 respectively), as compared with the students who used square-counting and calculation with formulas (.5 and 0 respectively). It is also worth noting that five students who were generally weak in mathematics considered the problem to be easier than usual.

In the interviews conducted after the intervention, the students were asked to explain the judgments they had made. For example, student T said:

Answer: The problems we usually solve in the textbook are more difficult.
Question: What is it that makes them more difficult?
Answer: Hmm... when I do not understand, this seems difficult.

Also, student A considered the problem to be easier, because "it didn't need many calculations and the like", and student C agreed also because "we were more students and we collaborated". On the contrary, student P, for example, thought that the problem was more difficult, because "it was more complicated", while student W, who had tiled equal surfaces, regarded the problem as more difficult, because "it puzzled you with the shapes, if it fits, if it leaves a gap, if you must... if you had to put something else".

Regarding interest, 12 students referred explicitly and clearly to nature, bees, honeycombs or to ancient Greeks and, more generally, to what constitutes the context of the problem, for example:

- "We learned many things about geometry, many ways to find the area and the perimeter of a shape, but we also learned about reality, why bees use this shape". (student I)
- "You were curious to see it; it is about nature and... it is a mystery what bees do, whereas the textbook's problems are, let's say, simpler". (student Q)
- "I liked it with the example we did, that is with bees and honeycombs and the text saying... It was like a story that you had to solve". (student M)

On the other hand, student B explicitly linked difficulty with interest: "It was more difficult; it was interesting to solve". There was also one mention of the fact that mathematicians worked on this problem and one answer saying that this way the students learned "how geometry was discovered" (student W), three mentions of the fact that the students worked in groups and two mentions of the fact that the problem was unusual; student X, for example, gave this characteristic answer: "I hadn't done a problem like this before and this is why I liked it".

4. DISCUSSION

In the present research, we used the history of mathematics to achieve various goals, which were related to each other and to the development of the students' GWSs as well. In particular, the historical sources were the source of geometric problems and problems are the reason of existence of GWSs (Kuzniak 2015). In addition, the historical sources and the historical note served as a means of motivation and motivation constitutes an important tool for the active involvement of students in solving problems (Brousseau 2002). Since the three texts were assessed positively, it could be argued that they all helped to motivate the students. Thus, the argument that many students may be affected negatively because they dislike history (Jankvist 2009, Tzanakis et al. 2000) was not supported here.

Pappus' text was also used as a means of activating preexisting definitions and properties of the theoretical frame of reference (e.g. definition of perimeter) and of enriching it with new definitions and properties (e.g. definition of regular figure); these properties were necessary for the development of the students' personal GWSs and the solution of the honeycomb problem. Additionally, the students made a first acquaintance with a new property concerning area-perimeter relationships. This property, however, was regarded by some students not as a proposition to be confirmed, but as established knowledge. This behaviour could be attributed to the usual didactic contract, according to which textbooks and, by extension, texts used in school, contain indisputable truths; it is also likely to reflect a broader conception according to which mathematical knowledge is generally unchanging over time (Schommer-Aikins 2002), and, therefore, a mathematician cannot be mistaken.

As already mentioned, the historical sources were the source of geometric problems. Subsequently, the honeycomb problem constituted a means of enriching the students' personal GWSs with new tools (transparent grid) as well as with experimentation methods which present area as an attribute and which had been used in previous grades but had been forgotten.

Furthermore, mathematical problems are a means to overcome students' misconceptions (Brousseau 2002), and Polybius' text seems to have contributed to this goal. This text is not a refutation text written for teaching purposes, but a historical source, with all its complexity. However, the reference to misconceptions related to area-perimeter relationships and the information that such mistakes were also made by important persons in history both acted as stimuli to the students and contributed to a climate of comfort for the students to reflect and talk about themselves. This is also related to the students' personal GWSs and, in particular, to their theoretical frame of reference.

Here, however, we should take into account that changing ideas is difficult and no text alone is sufficient to achieve this with all students (Tippett 2010). This is also true for area-perimeter relationships, in which even secondary school students and adults have difficulties (Kellogg 2010, Woodward & Byrd 1983). Besides, it has been observed that misconceptions concerning area-perimeter relationships often reappear after instruction (Douady & Perrin-Glorian 1989, Kellogg 2010, Vighi 2010).

Apart from these, it is interesting that two students spontaneously referred to their attitude towards geometry, although this was not the main goal of the intervention.

Regarding the assessment of the historical texts, the students on average answered that they liked all three texts; most of all they liked the extract from Pappus, then the extract from Polybius and finally the historical note. The difference was statistically significant in comparing the extract from Pappus with the historical note. These findings have multiple interpretations:

- The ranking of the three texts reflects the time allotted to each one. However, if the students did not like the way that teaching time was used, then more allotted time would have probably led to a greater dislike of a text.
- The students' greater preference for both primary sources is in accordance with the recommendation made by Pintrich & Schunk (2002) that original source material should be used. At the same time, this preference could be attributed to the fact that both primary sources were accompanied by a mathematical problem, whereas the historical note was not.
- The extract from Pappus was read aloud by the teacher, and this probably facilitated comprehension and made the text more vivid, thereby increasing the students' interest (Ariail & Albright 2006, Ivey & Broaddus 2001, Schraw et al. 1995).
- Most of all, it seems that the students' greater preference for the extract from Pappus is related to the theme and, generally, to the features of the text: regularity in nature, and the society of bees are two themes that had attracted the interest of philosophers and mathematicians since antiquity and were widely known (Cuomo 2000). Thus, we could say that these themes could be listed among those themes that are related to nature and are reported to stimulate interest (Bergin 1999). Besides, as student Q said: "it is about nature and... it is a mystery what bees do". Apart from this, Pappus' text was written as a literary introduction to his book with the aim of stimulating interest. And finally, the chosen extract does not contain names, and dates or numbers, unlike the other two texts.

Regarding the degree of difficulty of the honeycomb problem as compared with the usual problems, the students' opinions were not homogeneous: somewhat more students (11) considered the problem to be easier, whereas eight of the 22 said that it was more difficult. The students' opinions were influenced, first, by the method with which each student worked. Second, it seems that the students who were generally weak in mathematics considered the problem to be easier than usual, taking into account the lack of calculations, the material form of the shapes, the availability of tools appropriate for work within GI and the fact that the students worked in groups. Thus, they were able to participate and contribute to the solution, and it is indicative that in the group which used calculations with formulas the most difficult reconfiguration of the hexagon was performed by a student who was generally weak in mathematics. Furthermore, the (indirect) subdivision of the problem through the questions included in the accompanying worksheet is also likely to have helped those who face difficulties in organizing the problem solving process.

On the other hand, 21 out of the 22 students considered the problem to be more interesting than the usual problems. This finding, combined with the answers regarding the degree of difficulty, suggests a sufficient balance between the requirements of the problem and the level of knowledge and skill of each student. As shown previously, a factor that contributed to this balance was the hands-on nature of the activity. In addition, the arguments of the students show that group work, the unusual nature of the problem and, most of all, the context of the problem also stimulated interest. All these factors have been reported in the related literature (Bergin 1999, Mitchell 1993, Pintrich & Schunk 2002) and may have influenced the students' views both directly and indirectly. For example, group work affected the students not only directly, but also indirectly by facilitating the solution of the problem, thus intervening in the relationship between challenge and skill. Finally, the context of the problem was determined by Pappus' text, and it seems that the combination of the problem with the text linked knowledge with the questions that gave birth to it and gave meaning to the activity.

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APPENDIX

Figure 1: The isoperimetric figures used in the activity « Neighborhoods of Thessaloniki ».

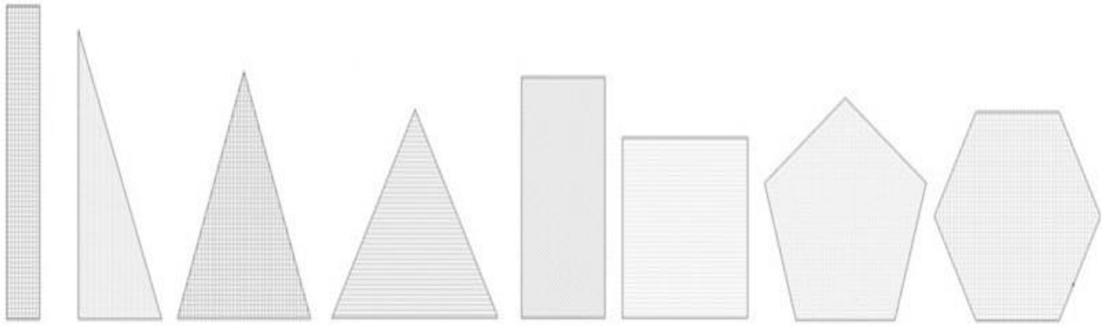

Figure 2: Superposition-reconfiguration; comparison between the hexagon and the square.

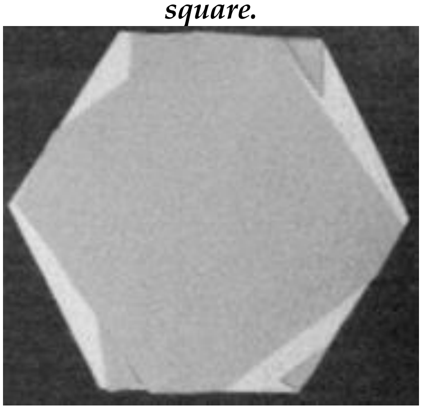

Figure 3: Tiling with regular hexagons.

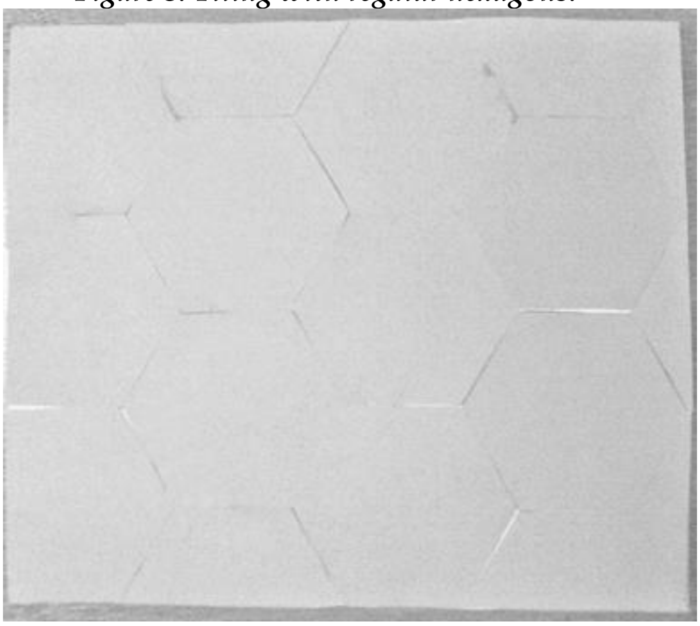

Figure 4: Square-counting in the case of the regular hexagon.

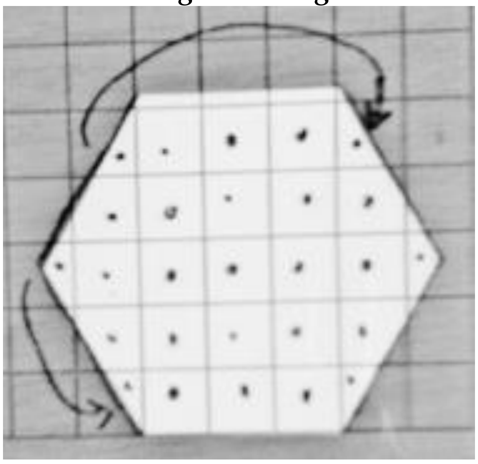

Figure 5: Reconfiguration of the equilateral triangle into a rectangle (4th method).