Ἔχεις μοι εἰπεῖν, ὦ Σώκρατε, ἃρα διδακτὸν ἢ ἄρετή; ἢ οὐ διδακτὸν ἄλλ’ ἀσκητὸν; ἢ οὖτε ἀσκητὸν οὐτε μαθητὸν, ἀλλὰ φύσει παραγίγνεται τοῖς ἀνθρώποις ἢ ἄλλω τινὶ τρόπων.
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The Editor and the Editorial Board of the **MENON: Journal Of Educational Research** thanks the following colleagues for their support in reviewing manuscripts for the current issue.

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- Konstantinos Christou
- Anna Chronaki
- Despina Desli
- Sonia Kafousi
- Fragiskos Kalavasis
- Eygenia Koleza
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EDITOR'S INTRODUCTORY NOTE

INTRODUCTION TO SPECIAL ISSUE

Behaviour of students, teachers and future teachers in mental calculation and estimation

We are happy to present the first Special Issue of our new journal “MENON: Journal for Educational Research” which was introduced in 2012. Research in Mathematics Education is a significant area of educational research, which is included in the topics of this journal.

“Behaviour of students, teachers and future teachers in mental calculation and estimation” has been chosen as the subject for this special issue on the ground of a number of reasons which are presented below.

Over the past decades, many studies have been conducted in the field of mental calculations and estimation and more precisely in relation to the definition of these concepts, the identification of the strategies used by various age groups, the relationship with other concepts, such as number sense, the procedural and conceptual understanding among others.

Many educational systems have updated the teaching of numbers and operations in mathematics, incorporating mental calculations and estimations in their elementary and middle education curricula.

Nowadays, it is considered timely to conduct research in the implementation of the teaching of mental calculation and estimation with whole and rational numbers as well as the recording of students' behaviour and the training of pre-service and in-service teachers in these concepts.

During the last decade, researches on mental computation and estimation with rational numbers has been conducted in the Laboratory of “Nature and Life Mathematic” at the University of Western Macedonia, some of which are presented in this issue.

Most of the papers included in this issue, refer to mental calculations and estimations with rational numbers, a topic that is not very common in the literature and covers a wide age range including elementary school students, adults, as well as pre- and in-service teachers. The researches are presented according to the age range of the participants.

- In their study Peters, De Smedt, Torbeyns, Ghesquière, Verschaffel distinguish between two types of strategies for subtraction: (1) direct subtraction, and (2) subtraction by addition, and provide an overview of the results of 5 related studies using non-verbal methods to investigate the flexible use of these strategies in both single- and multi-digit subtraction. Adults, students and
elementary school students with mathematical learning disabilities have participated in this research.

- **Anestakis and Lemonidis** in their study, investigate the computational estimation ability of adult learners and implement a teaching intervention about computational estimation in a Junior High School for Adults. They suggest incorporating computational estimation into Second Chance Schools and into adult numeracy teaching practices in general.

- The two papers of **Lemonidis, Nolka, Nikolantonakis** and **Lemonidis, Kaiafa** examine the behaviour of 5th and 6th grade students in computational estimation and in mental calculations with rational numbers, respectively. In these studies, the relation between students’ performance in computational estimation and mental calculations with rational number and problem solving ability are also examined.

Four studies on this issue, refer to the prospective elementary teachers' behaviour in mental calculation and estimation.

- **Anestakis and Desli** examined 113 prospective primary school teachers’ views of computational estimation and its teaching in primary school. Results revealed that the majority of prospective teachers identified the importance of computational estimation for both daily life and school.

- In their research **Kourkoulos and Chalepaki** interviewed and examined through a test 69 pre-service teachers aiming to investigate the factors that contribute to their computational estimation ability. They found five factors that contribute to computational estimation, such as the mathematical background and the attitude towards mathematics.

- **Lemonidis, Tsakiridou, Panou and Griva** used interviews to examine the knowledge and the strategy use of 50 pre-service teachers in multiplication tables and their mental flexibility in two-digit multiplications by using the method choice / no-choice by Lemaire & Siegler, (1995). The results showed that prospective teachers are not flexible in two-digit multiplications and their flexibility to mentally calculate two-digit multiplications is associated with their knowledge in prep and their prep response time.

- **Koleza and Koleli** have used a test to study the mental computations and estimation strategies of 87 pre-service teachers. The data revealed that the prospective teachers’ number sense concerning rational numbers, basic concepts of the decimal system and elementary numerical properties was very weak.

- **Lemonidis, Mouratoglou and Pnevmatikos** studied 80 in-service teachers’ performance and strategies in computational estimation and individual
differences concerning their age and work experience, their attitude towards mathematics and their prior performance in mathematics during high school years.

- The last paper of Lemonidis, Kermeli and Palaigeorgiou propose a teaching intervention to sixth grade students in order to promote understanding and enrich their conceptual strategy repertoire to carry out mental calculations with rational numbers. At the same time, three teachers’ attitudes towards teaching mental computation with rational numbers, were examined.

Finally, I would like to thank all the researchers from Belgium and Greece who contributed with their papers in this thematic issue, the colleagues from the laboratory of "Nature and Life Mathematics", the reviewers of the papers and Elias Indos for the organizational and technical support in the journal.

The Editor of the first Special Issue of “MENON: Journal for Educational Research”

Charalampos Lemonidis
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USING ADDITION TO SOLVE SUBTRACTION PROBLEMS IN THE NUMBER DOMAIN UP TO 20 AND 100

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**Abstract**

Over the past 4 decades, a lot of research has been done on the strategies that are used to mentally solve subtraction problems. In the present manuscript, 2 types of strategies are distinguished: (1) direct subtraction, in which the smaller number is directly subtracted from the larger one (e.g., $75 - 43 = 32$ by doing $75 - 40 = 35$, $35 - 3 = 32$), and (2) subtraction by addition, in which one determines how much needs to be added to the smaller number to get to the larger one (e.g., $75 - 43 = 32$ by doing $43 + 30 = 73$ and $73 + 2 = 75$, so the answer is $30 + 2 = 32$). We set up a series of 5 related studies using non-verbal methods to investigate the flexible use of these strategies in both single- and multi-digit subtraction, first in adults and then in both typically developing children and children with mathematical learning disabilities. The present article provides an overview of the results of these 5 studies.

**Keywords:** mental arithmetic, direct subtraction, subtraction by addition, flexible strategy use.

1. **Introduction**

In the last four decades, a worldwide reform movement has changed some of the founding principles of elementary mathematics education. According to these
reform-based ideas, instruction should no longer focus on solving school mathematics exercises quickly and accurately by means of the school-taught standard strategies (i.e., routine expertise), but children should solve mathematical tasks efficiently, creatively, and flexibly with a variety of meaningfully acquired strategies (i.e., adaptive expertise) (e.g., Baroody & Dowker, 2003; Hatano, 2003; Kilpatrick, Swafford, & Findell, 2001; Verschaffel, Greer, & De Corte, 2007). Although this idea of flexibility is also endorsed in the current attainment targets of mathematics education in Flanders (Vlaams Ministerie van Onderwijs en Vorming, 2010), Flemish publishers of mathematics textbooks and elementary school teachers still seem to value the fast and accurate execution of one strategy over the flexible use of different (self-invented) strategies. Moreover, there is still a lot of discussion whether strategy variety and flexibility should also be seen as important goals for low-achieving children (e.g., Baroody, 2003; Kilpatrick et al., 2001; Threlfall, 2002; Verschaffel, Torbeys, De Smedt, Luwel, & Van Dooren, 2007).

One of the mathematical subdomains in which strategy variety and flexibility can be especially aimed for and stimulated, is mental subtraction. The way children and adults solve symbolically presented subtraction problems of the type $M - S = \cdot$ (with $M = \text{minuend}$ and $S = \text{subtrahend}$) has consequently received a lot of research attention (e.g., Barrouillet, Mignon, & Thevenot, 2008; Fuson, 1992; Kraemer, 2009; Robinson, 2001; Selter, 2001; Torbeys, De Smedt, Ghesquière, & Verschaffel, 2009a; Woods, Resnick, & Groen, 1975). These studies have shown that people develop various strategies to mentally solve subtraction problems. While researchers assumed for a very long time that for solving single-digit subtraction problems (e.g., $9 - 2 = \cdot$ or $13 - 7 = \cdot$) children gradually move from counting-based strategies over procedural strategies to the (adult way of) direct retrieval of an answer from long-term memory, various studies from the last two decades showed that even adults often use non-retrieval strategies to solve subtraction problems (Ashcraft, 1992; Campbell & Xue, 2001; Geary, Frensch, & Wiley, 1993; LeFevre, DeStefano, Penner-Wilger, & Daley, 2006; Robinson, 2001; Seyler, Kirk, & Ashcraft, 2003). While retrieval seems to be mainly used for small single-digit subtraction problems (with minuend ≤ 10), large single-digit subtraction problems (with minuend between 10 and 20) are often solved by non-retrieval strategies based on counting (e.g., solving $11 - 2 = \cdot$ by counting down 2 starting from 11: “11, 10, 9”) or on the use of derived facts (e.g., solving $14 - 6 = \cdot$ by doing $14 - (4 + 2) = 10 - 2 = 8$).

In the domain of multi-digit subtraction, it has also been shown that adults and children develop various strategies to solve subtraction problems (e.g., Beishuizen, 1993; Blöte, Klein, & Beishuizen, 2000; Carpenter, Franke, Jacobs, Fennema, & Empson, 1997; Heirdsfield & Cooper, 2004; Selter, 1998; Thompson, 2000; Verschaffel, Greer, et al., 2007). Most of these studies categorised strategies into three main types, based on the way the subtraction operation is executed. In the first type, the split or decomposition strategies, the tens and the units of both minuend and subtrahend are split, and then these tens and units are subtracted separately
from each other (e.g., 75 − 43 = . by doing 70 − 40 and 5 − 3, so the answer is 30 + 2 = 32). In the second type, the jump or sequential strategies, the minuend is kept as a whole and a problem is solved by subtracting the tens and the units of the subtrahend in two distinctive steps from the minuend (e.g., 75 − 43 = . by doing 75 − 40 = 35, and 35 − 3 = 32). The third type of subtraction strategies, varying strategies, involves the flexible adaptation of the numbers and/or operations in the problem based on people’s understanding of the number relations or the properties of the arithmetic operations. An example of such a varying strategy is the compensation strategy, in which one or both of the numbers are changed to make the computations easier (e.g., 75 − 43 = . by doing 75 − 45 = 30, and then adding 2 because of the compensation of the original subtrahend, so the answer is 30 + 2 = 32).

Most interesting within the context of the present manuscript is the observation that in both number domains (i.e., single-digit and multi-digit subtraction) adults and children sometimes solve subtraction problems by using an addition operation (Baroody, Torbecyns, & Verschaffel, 2009; Campbell, 2008; Fuson & Willis, 1988; Geary et al., 1993; Kraemer, 2009; Menne, 2001; Seyler et al., 2003; Torbecyns, De Smedt, Ghesquière, et al., 2009a; Torbecyns, Ghesquière, & Verschaffel, 2009; Verschaffel, Bryant, & Torbecyns, 2012; Woods et al., 1975). They report, for instance, that they know the result of a problem such as 7 − 3 = . because 3 + 4 = 7, they sometimes count on from the smaller to the larger number (e.g., solving 81 − 79 = . by doing “79... 80, 81, so the answer is 2"), or they report that they solve a problem such as 75 − 43 = . by asking themselves how much needs to be added to the smallest number to get to the largest one, for example by doing 43 + 2 = 45 and 45 + 30 = 75, so the answer is 2 + 30 = 32. Consequently, a second type of classification of subtraction strategies is possible, by looking at the operation that underlies the solution process. In this way, two types of strategies can be distinguished: (1) direct subtraction strategies, in which the subtrahend is directly subtracted from the minuend (e.g., 75 − 43 = . by 75 − 40 = 35, 35 − 3 = 32), and (2) subtraction by addition strategies, in which one determines how much needs to be added to the subtrahend to get to the minuend (e.g., 75 − 43 = . by 43 + 30 = 73 and 73 + 2 = 75, so the answer is 30 + 2 = 32).

Over the years, several studies at the Leuven Centre for Instructional Psychology and Technology (CIP&T) have investigated the use of the subtraction by addition strategy on symbolically presented subtraction problems. They used Lemaire and Siegler’s (1995) model of strategy change and strategy choice as a theoretical framework, and particularly focused on the parameter that refers to the adaptiveness or flexibility of someone’s strategy choices. In this model of strategy change and strategy choice, someone’s strategy choice is called flexible if (s)he chooses the strategy from his/her strategy repertoire that will lead fastest to an accurate answer. Through practicing several similar problems over time, someone will learn to use more efficient strategies, which will be executed more frequently, more efficiently, and also more adaptively.
Direct subtraction and subtraction by addition are two strategies for which an adaptive strategy choice based on task characteristics can be particularly efficient (e.g., Torbeyns, De Smedt, Stassens, Ghesquière, & Verschaffel, 2009): Based on a rational task analysis, direct subtraction is assumed to elicit few and/or small counting/calculation steps when the subtrahend is relatively small compared to the difference (such as 9 − 2 = ., 12 − 3 = . or 81 − 2 = .), but more and/or larger steps when the subtrahend is relatively large compared to the difference (such as 9 − 7 = ., 12 − 9 = . or 81 − 79 = .). Following the same logic, the opposite process is expected to happen for the subtraction by addition strategy: Few and/or small counting/calculation steps are needed when the subtrahend is relatively large (such as 9 − 7 = ., 12 − 9 = . or 81 − 79 = .), but more and/or larger steps when the subtrahend is relatively small compared to the difference (such as 9 − 2 = ., 12 − 3 = . or 81 − 2 = .). Selecting the strategy for which the counting/calculation steps are very small and easy can be seen as particularly efficient, because this quick and easy counting/subtraction process will very often also lead to a correct answer. Compare, for example, solving 81 − 79 = . by means of direct subtraction (e.g., 81 − 70 = 11 and 11 − 9 = 10 − 8 = 2) with using the subtraction by addition strategy to solve it (e.g., 79 + 1 = 80 and 80 + 1 = 81, so the answer is 1 + 1 = 2).

While these previous studies have shown that adults use the subtraction by addition strategy frequently, efficiently, and flexibly (i.e., mainly, but not exclusively, on problems with a relatively large subtrahend) (e.g., Torbeyns, Ghesquière, et al., 2009; Torbeyns, De Smedt, Peters, Ghesquière, & Verschaffel, 2011), well-documented evidence of elementary school children’s use of the subtraction by addition strategy is very scarce (e.g., De Smedt, Torbeyns, Stassens, Ghesquière, & Verschaffel; 2010; Torbeyns, De Smedt, Ghesquière, et al., 2009a, 2009b). For example, Torbeyns, De Smedt, Ghesquière, et al. (2009a) asked Flemish second-, third-, and fourth-graders to solve two-digit subtractions in two tasks. In the Spontaneous Strategy Use Task, children were asked to solve each problem as fast and as accurately as possible with their preferred strategy, and to verbally report both the answer and the strategy they used after solving each problem. Five out of the 15 presented two-digit subtraction problems had a relatively large subtrahend (as in 81 − 79 = .), which was assumed to trigger the subtraction by addition strategy. Still, children hardly applied this strategy spontaneously: Second- and third-graders did so in about 5 % of the cases on the five problems with a relatively large subtrahend; fourth-graders in only 9 % of these cases. In the Variability on Demand Task, children were asked to generate up to five different strategies for solving 4 two-digit subtraction problems. Now, two of these problems were assumed to trigger the subtraction by addition strategy. Surprisingly, the frequency of subtraction by addition strategies did not differ that much from the first task: Only 4 % of the second-graders, 13 % of the third-graders, and 21 % of the fourth-graders reported subtraction by addition as a possible way to solve the two problems. The authors
concluded that the subtraction by addition strategy was not included in most children’s strategy repertoire.

Similar results were reported by Torbeyns, De Smedt, Ghesquière, et al. (2009b) when comparing two groups of second- to fourth-graders who were asked to write down their solution steps when solving symbolically presented two-digit subtraction problems. These two groups only differed in the received instruction about the subtraction by addition strategy. In one group no instruction about this strategy was given, whereas the second group had followed a mathematics textbook from first grade on that focused on subtraction by addition as the alternative for the direct subtraction strategy for problems with a relatively small difference⁴. Both groups of children were asked to solve the same 16 symbolically presented two-digit subtractions in whatever way they wanted, and to write down their solution steps. Half of the experimental items were designed with a difference smaller than 10, in order to elicit the use of subtraction by addition as much as possible. However, only 0.2 % of all written reports in the no-instruction group, but also no more than 7.5 % of all written solutions in the instruction-group, could be identified as subtraction by addition. Again, the authors concluded that the subtraction by addition strategy was hardly used, even by children who had received instruction about and practice in this strategy.

De Smedt et al. (2010) tried to stimulate the use of subtraction by addition in third-grade children. Participants were divided over an implicit and explicit learning environment, both of which involved four training sessions. In the implicit learning environment, children were confronted with an unusually large number of two-digit subtraction problems with a relatively large subtrahend. This was done because Flemish textbooks hardly contain this type of subtraction problems, although they are most suitable for discovering the computational advantage of the subtraction by addition strategy. Children in the explicit learning environment were instructed to solve each problem once with direct subtraction and once with subtraction by addition, which was explained at the beginning of each training session. None of the children from the implicit learning environment reported using the subtraction by addition strategy in the test session halfway the training, at the end of the training, or in the retention session one month later. In the explicit learning environment, only 6 % of the children reported using the subtraction by addition strategy in the test session after two training sessions, only 11 % reported subtraction by addition by the end of the training sessions, and only 10 % reported it one month later. From these low percentages, the authors inferred that – even in the explicit learning environment – third-grade children experienced great difficulties with picking up and integrating the subtraction by addition strategy into their strategy repertoire.

2. FIVE RELATED STUDIES ON THE FLEXIBLE USE OF THE SUBTRACTION BY ADDITION STRATEGY

One important limitation of the studies reviewed above is that they relied exclusively on verbal or written data to detect the subtraction by addition strategy. These
methods might, however, not be the best way to identify certain types of mental calculation strategies (e.g., Cooney & Ladd, 1992; Ericsson & Simon, 1993; Kirk & Ashcraft, 2001; LeFevre, Smith-Chant, Hiscock, Daley, & Morris, 2003; LeFevre, Sadesky, & Bisanz, 1996; Russo, Johnson, & Stephens, 1989; Siegler & Stern, 1998). Kirk and Ashcraft (2001), for example, wrote that the validity of using verbal protocols can be questioned when studying certain mathematical processes, a statement they base on problems with veridicality, reactivity, and demand. First, they argue that the mental processes that underlie a solution process cannot always be accurately reflected on in verbal reports, especially when the execution of the solution process is automatic (i.e., not involving working memory processing). When solving a problem such as 12 − 9 = . or 81 − 79 = . (i.e., problems for which the subtraction by addition strategy seems to be very efficient), people may thus not be aware of the calculation steps they executed, and only have access to the outcome of the problem. Strategy reports for these problems might therefore not be veridical to what really happened when solving the problem. Secondly, according to Kirk and Ashcraft, asking participants how they solve a problem might change the mental processes occurring in normal settings (see also Ericsson & Simon [1993] and Russo et al. [1989]): It might be that participants are pressured to perform better in terms of accuracy since verbal strategy reports seem to be a platform that explicitly shows the errors they make, which forces them into changing their normal routines. Similarly, they might even deliberately change the strategies they would spontaneously use because they know they might not be able to explain them and therefore choose to select strategies from their strategy repertoire which they do have the words for. In other words, they might react to the setting that they are put in when participating in an experiment.

As a third problem, Kirk and Ashcraft discuss the possibility that participants might figure out the goal of an experiment and therefore state answers or processes that they think the experimenters are interested most in. Participants might thus deliberately hide the use of a particular strategy because they think it is not valued or even not allowed, and therefore report the strategies they think the setting demands. Especially in children - who are in their classroom often confronted with only one strategy to solve certain types of problems and therefore might have the idea that other strategies are not allowed (socio-mathematical classroom norms, Yackel & Cobb [1996]) - this might be a very plausible reaction.

When taking these possible methodological problems into account, the results of previous research on children’s use of the subtraction by addition strategy might represent an underestimation of their use of this strategy. In this respect, we point to an inconsistency between the verbal reports and children’s reaction time data in De Smedt et al. (2010). The vast majority of the third-graders in this study only reported to use the direct subtraction strategy. However, if this had actually been the case, there should have been an increase in children’s reaction times from problems with relatively small subtrahends (e.g., 81 − 7 = .) over problems with medium-sized subtrahends (e.g., 81 − 43 = .) to problems with relatively large subtrahends (e.g., 81...
− 79 = .), because according to the above-mentioned rational task analysis, subtracting a larger subtrahend requires more and/or larger calculation steps. The observed reaction time patterns, however, argue against this interpretation: not only problems with relatively small but also problems with relatively large subtrahends were solved faster than problems with medium-sized subtrahends. These reaction time data thus suggest that the verbal report data were not always in line with the strategies actually applied by these children. More specifically, they indicate that the subtraction by addition strategy might have been used more frequently than suggested by the children’s verbal reports.

We therefore aimed at studying the use of subtraction by addition with other, non-verbal methods for inferring strategy use. More particularly, we applied non-verbal methods for investigating the flexible use of subtraction by addition in both single- and multi-digit subtraction, first in adults and then in both typically developing children and children with mathematical learning disabilities. Here we will present a short overview of all five studies; for more details we refer to the original manuscripts (Peters, 2013; Peters, De Smedt, Torbeyns, Ghesquière, & Verschaffel, 2010a, 2010b, 2012, 2013; Peters, De Smedt, Torbeyns, Verschaffel & Ghesquière, 2014).

3. STUDY 1

In Study 1 (Peters et al., 2010a), we investigated 25 university students’ (Mean age = 26 years; SD = 7 years) use of the subtraction by addition strategy when solving large single-digit subtraction problems (i.e., problems with a minuend larger than 10 and a single-digit subtrahend). We extended the work of Campbell (2008), who presented students with subtraction problems in the standard subtraction format (9 − 2 = .) and in their corresponding addition format (2 + . = 9) and who studied the effect of these different presentation formats on adults’ reaction times. Campbell found that problems presented in the addition format were solved faster than those in subtraction format and he concluded that large single-digit subtractions are often solved by means of addition.

However, previous research on related arithmetic tasks (e.g., Brissiaud, 1994; Torbeyns, Ghesquière et al., 2009; Woods et al., 1975) suggests that this use of subtraction by addition may depend on the relative size of the subtrahend, a task parameter that was not systematically addressed in Campbell (2008). We therefore extended Campbell’s research method and tested whether students solved large single-digit subtractions by flexibly switching between the direct subtraction and the subtraction by addition strategy, depending on which of both processes requires the fewest or smallest steps. We compared solution times on the 32 large non-tie subtraction problems presented once in the standard subtraction format (12 − 9 = .) and once in an addition format (9 + . = 12). We systematically manipulated the relative size of the subtrahend in these problems by combining two problem characteristics: the magnitude of the subtrahend (S) compared to the difference (D) (S < D vs. S > D), and the numerical distance between S and D (small-distance
problems were defined by $S$ and $D$ differing by only 1 or 2, whereas in the large-distance problems $S$ and $D$ differ by more than 2).

All participants performed a computer task in which they had to solve a total of 64 items. Each trial started with an asterisk that appeared for 1000 ms in the centre of the screen. Next, the item was presented (horizontally) in the middle of the screen. Participants’ solution time started to run when the item appeared on the screen, and ended when a sound was detected. We performed a repeated measures ANOVA on the solutions times with magnitude of the subtrahend ($S < D$ vs. $S > D$), numerical distance (large vs. small), and format (subtraction vs. addition) as within-subject factors. We found a significant three-way interaction: When the subtrahend was larger than the difference and $S$ and $D$ were far from each other (e.g., $12 - 9 = .$), problems were solved faster in the addition than in the subtraction format; when the subtrahend was smaller than the difference and $S$ and $D$ were far from each other (e.g., $12 - 3 = .$), problems were solved faster in the subtraction than in the addition format. However, when the subtrahend and the difference were close to each other (e.g., $13 - 6 = .$ and $15 - 8 = .$), there were no significant reaction time differences between both formats. These results suggest that adults do not rely exclusively on addition to solve large single-digit subtractions, but select either direct subtraction or subtraction by addition, depending on the relative size of the subtrahend.

4. STUDY 2

In Study 2 (Peters et al., 2010b), we extended our research question to the number domain up to 100. Several authors reported adults using the subtraction by addition strategy in multi-digit subtraction. Starting from the results of Torbeyns, Ghesquière, et al. (2009), we hypothesised that we would again find a flexible strategy choice pattern between direct subtraction and subtraction by addition, based on the relative size of the subtrahend. Twenty-five university students (Mean age = 26 years; $SD = 7$ years) were asked to solve 32 two-digit subtraction problems, once presented in the standard subtraction format ($81 - 37 = .$) and once in an addition format ($37 + . = 81$). We first calculated a stepwise regression model – based on Groen and Poll (1973) and Woods et al. (1975) in the domain of single-digit arithmetic – in which participants’ reaction times on two-digit subtractions were predicted by the presentation format ($M - S = .$ or $S + . = M$), three variables referring to strategy use (the to-be-determined difference $[D]$, the known subtrahend $[S]$, and the minimum of difference and subtrahend (min$[D, S]$)), and their respective interactions. We expected that the model including the min ($D, S$) predictor as the variable that explains most of the variability in reaction times would provide the best fit, suggesting the mixed use of subtraction by addition and direct subtraction in both presentation formats. This was indeed the case.

Second, we performed exactly the same analysis as in Study 1. The 32 two-digit problems were divided into four problem types, based on the combination of the magnitude of $S$ compared to $D$ ($S < D$ or $S > D$) and the numerical distance between $S$
and $D$ (small or large). Small-distance problems were defined by $S$ and $D$ differing by less than 10, whereas in the large-distance problems $S$ and $D$ differed by at least 10 and either $S$ or $D$ was a one-digit number. We again compared reaction times on the problems presented in the two presentation formats, and found that students switched between direct subtraction and subtraction by addition depending on the relative size of the subtrahend: If the subtrahend was smaller than the difference (e.g., $83 - 4 = .$), direct subtraction was mainly used; if the subtrahend was larger than the difference (e.g., $83 - 79 = .$), subtraction by addition was the dominant strategy. However, this performance pattern was only observed when the distance between the subtrahend and the difference was large; when the subtrahend and the difference were close to each other (e.g., $81 - 37 = .$ or $81 - 44 = .$) there was no subtrahend-dependent selection of direct subtraction vs. subtraction by addition. The results of Study 2 thus indicate again that the relative size of the subtrahend is an important factor in the strategy selection process.

5. STUDY 3

In Study 3 (Peters et al., 2012), we focussed on primary school children solving large single-digit subtraction problems (i.e., problems with a minuend larger than 10 and a single-digit subtrahend). Whereas several studies showed that children seem to apply the subtraction by addition strategy when confronted with small single-digit subtraction problems (with a minuend $\leq 10$) (e.g., Barrouillet et al., 2008; Carpenter & Moser, 1984; Fuson, 1992), there is hardly any research concerning children’s flexible use of the subtraction by addition strategy on large single-digit subtraction problems. Therefore, we tested this issue through a replication of our study on adults’ use of subtraction by addition on large single-digit subtraction (Study 1) in elementary school children, now using regression-based analyses.

We presented 106 third- to sixth-graders (mean ages were 8 years 11 months [$SD = 3$ months] in third grade, 9 years and 11 months [$SD = 4$ months] in fourth grade, 10 years and 10 months [$SD = 4$ months] in fifth grade, and 11 years and 10 months [$SD = 3$ months] in sixth grade) with the 32 non-tie subtraction problems, once in the standard subtraction format ($12 - 9 = .$) and once the addition format ($9 + . = 12$). For both presentation formats separately, we compared the fit of three linear regression models, which represented, respectively, the consistent use of direct subtraction, of subtraction by addition, and of flexibly switching between both strategies based on the relative size of the subtrahend. The first model, the $DS$-Model, represented children only using the direct subtraction strategy. If this was the case, their reaction times should be best predicted by the size of the known subtrahend ($S$), because it takes longer to subtract 9 from a given number than to subtract 3 from that number. In the second model, the $SBA$-Model, the consistent use of the subtraction by addition strategy was represented. According to this model, children’s reaction times should be best predicted by the size of the to-be-determined difference ($D$), because it takes longer to determine how much needs to be added to get at a given number.
when the difference between both numbers is relatively large (“How much needs to be added to 3 to have 12?”) than when it is relatively small (“How much needs to be added to 9 to have 12?”). Finally, if children switched flexibly between both strategies depending on which strategy is most efficient, as represented by the Switch-Model, reaction times should be best predicted by the minimum of the subtrahend and the difference \((\min[D, S])\): For problems with the subtrahend smaller than the difference (e.g., \(12 - 3 = .\) and \(12 - 5 = .\)), we expect reaction times to increase with the size of the subtrahend, because these problems can be quickly solved by means of the direct subtraction strategy. In contrast, for problems with the difference smaller than the subtrahend (e.g., \(12 - 9 = .\) and \(12 - 7 = .\)), we expect reaction times to increase with the size of the difference, because these problems can be quickly solved by means of subtraction by addition. Findings revealed that children did not switch flexibly between the two strategies, as adults did in Study 1, but that they relied only on the direct subtraction strategy to solve the standard subtraction problems \((M - S = .)\) and only on subtraction by addition for problems presented in the addition format \((S + . = M)\), independently of the relative size of the subtrahend.

6. STUDY 4

In Study 4 (Peters et al., 2013), we replicated the study on adults’ use of the subtraction by addition strategy on two-digit subtraction problems (Study 2) in 72 fourth- to sixth-grade elementary school children (mean ages were 9 years and 9 months \([SD = 3\) months\]) in fourth grade, 10 years and 10 months \([SD = 4\) months\]) in fifth grade, and 11 years and 9 months \([SD = 3\) months\]) in sixth grade. We presented them with 32 two-digit subtractions, which could be classified into four problem types (see Study 2). First, we fitted the same three regression models as we did in Study 3 to the reaction times of these 32 two-digit subtractions. These models represented either the use of direct subtraction, subtraction by addition, and switching between the two strategies based on the magnitude of the subtrahend. Additionally, we compared reaction times on problems presented once in the standard subtraction format \((81 - 37 = .)\) and once in an addition format \((37 + . = 81)\), as we did in Studies 1 and 2. Both methods converged to the conclusion that children of all three grades switched between direct subtraction and subtraction by addition based on the combination of two features of the subtrahend: If the subtrahend was smaller than the difference (e.g., \(83 - 4 = .\)), direct subtraction was the dominant strategy; if the subtrahend was larger than the difference (e.g., \(83 - 79 = .\)), subtraction by addition was mainly observed. However, this performance pattern was only observed when the numerical distance between subtrahend and difference was large.

7. STUDY 5

Finally, we investigated the use of subtraction by addition in children with mathematical learning disabilities (MLD) (Peters et al., 2014). As stated before,
especially for these children the idea of stimulating strategy variability and flexibility is still subject to discussion among scholars (e.g., Baroody, 2003; Kilpatrick et al., 2001; Threlfall, 2002; Verschaffel, Torbeyns, et al., 2007). Some researchers and policy makers advise to teach MLD children only one solution strategy, others advocate stimulating the flexible use of various strategies, as for typically developing children. To contribute to this debate, we investigated the use of the subtraction by addition strategy to mentally solve two-digit subtractions in 44 children with MLD (mean age 12 years and 5 months [SD = 6 months]). We conducted a replication of Study 4, and thus again used two non-verbal research methods to infer strategy use patterns. First, we fitted three regression models to the reaction times of 32 two-digit subtractions. Additionally, we compared performance on problems presented in two presentation formats (e.g., 81 − 37 = . and 37 + . = 81). We found that MLD children – similar to their typically developing peers – switch between the traditionally taught direct subtraction strategy and subtraction by addition, based on the relative size of the subtrahend. These findings challenge typical special education classroom practices, which only focus on the routine mastery of the direct subtraction strategy.

8. CONCLUSION AND DISCUSSION

We have reported on five closely related studies on the use of the subtraction by addition strategy to solve symbolically presented subtraction problems. In all studies, we investigated whether adults (Studies 1 and 2) or elementary school children (typically developing children in Studies 3 and 4, and children with MLD in Study 5) switch between the subtraction by addition strategy and the direct subtraction strategy to solve subtraction problems, and whether they base their strategy choice on the relative size of the subtrahend. Based on previous work in the fields of cognitive psychology and mathematics education (e.g., Torbeyns, Ghesquière, et al., 2009; Woods et al., 1975), we hypothesized that this task characteristic would influence the strategy choice process because of its connection to the efficiency of the calculation process: Problems with a relatively small subtrahend (such as 12 − 3 = . or 81 − 2 = .) can be solved very fast and easy by taking away the subtrahend from the minuend, whereas for problems with a relatively large subtrahend (such as 12 − 9 = . or 81 − 79 = .) it can be faster and easier to determine the difference by adding on to the subtrahend to get to the minuend.

We focused on strategy use on symbolically presented problems in two mathematical domains, i.e., large single-digit subtraction (with minuends between 10 and 20) and two-digit subtraction. In all five studies, all problems involved borrowing and were presented in both the standard subtraction format (M − S = .) and the addition format (S + . = M). In four of the five studies participants based their strategy choices on the relative size of the subtrahend: For both presentation formats, they showed strategy use patterns which represent the use of direct subtraction when the subtrahend was relatively small and subtraction by addition
when the subtrahend was relatively large. These patterns were found for adults who solved large single-digit and two-digit subtractions, and for both typically developing children and children with MLD who solved two-digit subtractions. However, a different result was found for the typically developing children who solved large single-digit subtraction problems such as 12 – 9 = ? in Study 3: Reaction time patterns suggested that these children used the direct subtraction strategy when those problems were presented in the standard subtraction format \((M - S = \cdot)\), and subtraction by addition for problems presented in the addition format \((S + \cdot = M)\), independently of the relative size of the subtrahend.

How can we explain this latter contrasting finding? Since the results presented in Studies 4 and 5 showed that children flexibly switched between direct subtraction and subtraction by addition in the number domain up to 100, a lack of conceptual understanding of the addition/subtraction complement principle, the inability of inhibiting the taking-away interpretation of the minus sign, and the effect of classroom norms and practices through which flexible strategy use is not stimulated (i.e., three possible explanations given in Peters et al., 2012) seem to be invalidated. Two other explanations, namely (1) the way subtraction with borrowing at 10 is instructed in Flemish elementary schools and (2) the influence of a strategy switch cost (i.e., the two remaining explanations given in Peters et al., 2012) still seem valuable explanations for this contrastive finding. First, the instruction in solving large single-digit subtractions in Flemish elementary schools focuses primarily on splitting the subtrahend into two parts, in order to substitute the original problem into two easier problems involving the number 10 (Van Olmen, 2005). From first grade on, children practice this decomposition strategy so often that the confrontation with a large single-digit subtraction problem may automatically trigger the use of the direct subtraction strategy in that number domain. Second, and somewhat related to the first explanation, it might be that children do not switch between the two strategies because the possible advantages of such a switch do not compensate for the cognitive cost (in terms of time and effort) involved in making the strategy switch (e.g., Lemaire & Lecacheur, 2010; Luwel, Onghena, Torbeyns, Schillemans, & Verschaffel, 2009). They might have been so proficient in using the direct subtraction strategy on problems in subtraction format that merely the process of switching to an alternative strategy would involve a cognitive cost being larger than sticking to direct subtraction. Both remaining explanations should be addressed in more detail in future research.

The finding that both typically developing children and children with MLD flexibly choose between the two strategies is in contrast with previous research on children’s strategy use on symbolically presented two-digit subtraction problems (e.g., De Smedt et al., 2010; Torbeyns, De Smedt, Ghesquière, et al., 2009a, 2009b). As we stated in the introduction, in this previous research hardly any children reported to use the subtraction by addition strategy when they were asked to explain how they solved the problems. Based on the findings reported in the present overview, we
have to question why in previous research children did not report using the subtraction by addition strategy. Were they not aware of the calculation steps they had executed? Did they have difficulties in articulating precisely how they found the answer (and therefore reported the direct subtraction strategy, which they had learnt to verbalize during the numerous mathematics lessons wherein they had practiced that strategy)? Or, did they deliberately hide the use of the subtraction by addition strategy because they taught it was not valued, or even not allowed, to solve a subtraction problem in that way? All these explanations seem possible, and should be examined in more detail in further research. However, the two non-verbal research methods used in the five present studies have their limitations too (see Peters, 2013). Other research methods, such as using eye-movements, could therefore be used in future studies to further investigate and enhance the validity of both verbal and non-verbal research methods through triangulation.

At a more general theoretical level, our findings are in accordance with Siegler's SCADS* model (Siegler & Araya, 2005; see also Verschaffel, Luwel et al., 2009). This model contends that when confronted with cognitive tasks people make adaptive strategy choices and take into account knowledge about the efficiency of a particular strategy for a particular problem type. Four of the five present studies revealed that both adults and children rely on two task features when choosing between the direct subtraction and the subtraction by addition strategy: (1) the magnitude of the subtrahend, and (2) the numerical distance between subtrahend and difference (the combination of which has been termed by Peters (2013) and also in this article as the relative size of the subtrahend). Taking into account these two task characteristics involves some kind of comparison between the numbers in the problem. Arguably, this comparison occurs during the orienting or planning phase of the solution process and relies on fast (quasi-)automatic processes of estimation and number sense rather than precise calculations of differences between the numbers. Several studies have highlighted the importance of magnitude comparison skills and number sense for successful mathematical development (e.g., De Smedt, Verschaffel, & Ghesquière, 2009; Gilmore, McCarthy, & Spelke, 2007; Holloway & Ansari, 2009; Sasanguie, Van den Bussche, & Reynvoet, 2012; Vanbinst, Ghesquière, & De Smedt, 2012). Future research should investigate the role of these characteristics in the execution of strategies in both single- and multi-digit subtraction in more detail. Based on the results presented in Study 3, the role of the presentation format and the avoidance of strategy switch cost should also be further investigated. All this suggests that in the future not only the influence of task characteristics on strategy flexibility should be looked into, but subject characteristics should be included as well. In that way, theoretical models about people's strategy choices when solving symbolically presented subtraction problems can be further refined.

Finally, we discuss some implications of our results for mathematics education. First, we want to stress that additional questionnaires with teachers in Studies 3, 4, and 5 showed that in the majority of the participating schools the subtraction by
addition strategy was not systematically and explicitly taught as a valuable alternative for the direct subtraction strategy to solve symbolically presented subtraction problems. Still, according to our non-verbal methods, this strategy was part not only of typically developing but also of MLD children’s strategy repertoire. In the number domain up to 100 (presented in Studies 4 and 5) the subtraction by addition strategy was even used flexibly, based on the relative size of the subtrahend. Taking also into account that the children’s mathematical textbooks did not provide them with many opportunities to discover and practice this strategy – since the Flemish textbooks hardly contain problems with a relatively large subtrahend (De Smedt et al., 2010) – this result was quite surprising, and it shows that teachers should be aware that generally speaking children (both in normal and special education schools) can do more than they are taught to do and more than is expected from them.

Second, teachers, teacher trainers, and material developers should be made aware of the possible problems linked to asking children how they solved a problem, so they can pay attention to them in their practices. Taking into account the attainment targets for Flemish elementary mathematics education (Vlaams Ministerie van Onderwijs en Vorming, 2010) – in which strategy flexibility is emphasized as a core goal – we think our results show that strategy variety and flexibility need to be emphasized more in the mathematics classroom. For example, teachers can operationalize this core goal by giving more attention to classroom discussions on “the how, the when, and the why” of strategy use (as in Selter, 1998). Through such discussions, children may be confronted with and start to try out new strategies by listening to the solution processes of their peers. Moreover, in those discussions teachers can push their students to reflect upon the suitability of a strategy for a given type of problem, and stimulate them to make connections to underlying mathematical principles (such as the addition/subtraction complement principle for the subtraction by addition strategy). Finally, such classroom discussions can also result in more positive attitudes, beliefs and emotions towards strategy variety and flexibility in mathematics education in general (see also Verschaffel, Luwel, et al., 2009).

9. FOOTNOTES

1 People can also use a third strategy, the so-called indirect subtraction strategy in which they determine how much needs to be subtracted from the minuend to get to the subtrahend (e.g., 75 – 43 = . by 75 – 30 = 45 and 45 – 2 = 43; so the answer is 30 + 2 = 32) (De Corte & Verschaffel, 1987). This indirect subtraction strategy may be particularly efficient on problems with relatively large subtrahends (e.g., 81 – 79 = .). However, previous studies on people’s strategy use in subtraction revealed that participants use this strategy only very rarely or not even at all (Beishuizen, Van Putten, & Van Mulken, 1997; Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009b; Van Lieshout, 1997).
Several authors have reported the use of these addition-based strategies, but different terms are used to denote them, such as the forward strategy (Brissiaud, 1994), the adding-on-to strategy (Menne, 2001); solving subtractions by means of addition (Beishuizen, 1997), the short jump strategy (Blöte et al., 2000), the adding up strategy (Selter, 2001), or indirect addition (Torbeyns, Ghesquière, et al., 2009).

Although subject and context characteristics might also play an important role in strategy choice processes (e.g., Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009), we only included task characteristics to operationalize flexibility and adaptivity in this research project.

Children who learn to do math with the textbook “Vaardig en Vlot” (Lowagie & Staelens, 1998) are taught to climb the ladder (i.e., the name they use to describe subtraction by addition) for symbolically presented subtraction problems with minuends < 10 and the subtrahend larger than the difference (e.g., 9 − 5 = ., 8 − 6 = ., and 7 − 4 = .). When confronted with problems with minuends between 10 and 20, they continue to climb the ladder when the difference is small (e.g., 19 − 17 = . and 12 − 8 = .). When the minuend is between 20 and 100, children are taught to climb the ladder when the difference is smaller than 10 (e.g., 57 − 49 = . and 91 − 85 = .).

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**BRIEF BIOGRAPHY**

At the time of this research project, Greet Peters was a Research Assistant for the Research Foundation Flanders (Belgium). The research was partially supported by Grant GOA 2012/10 "Number sense: Analysis and improvement" from the Research Fund KU Leuven, Belgium.
COMPUTATIONAL ESTIMATION IN AN ADULT SECONDARY SCHOOL: A TEACHING EXPERIMENT

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ABSTRACT

The purpose of this study is to investigate the computational estimation ability of adult learners and to implement a teaching intervention about computational estimation in an Adult Junior High School. Adults’ estimation skill is measured through interview, before and after the intervention, with a researcher-designed tool. A questionnaire is also used to evaluate their attitudes towards mathematics. The math teacher who implemented the teaching experimental is interviewed too. The results suggest that, although adults could estimate to some extent, their estimation skill was significantly improved due to the teaching intervention. Factors that may affect estimation skill, such as prior knowledge of strategies and involvement in everyday activities with measurements or mental calculations, are also investigated. Furthermore, math teacher’s participation in this research may have led to a PCK expansion, even though her attitude towards estimation was neutral. Finally, educational implications are discussed too.

Keywords: computational estimation, adult learners, teaching experiment.

1. INTRODUCTION

Literature about estimation has covered many topics mainly focusing on computational estimation ability. As expected, most research findings concentrate on students’ strategies and performance. It seems that performance and estimation strategies can differ depending on educational settings or on culture. It is also known that estimation ability improves with age (Lemaire & Lecacheur, 2002). Although many papers investigate adults’ estimation ability as well, most samples involve teachers. Only few studies examined computational estimation ability of adult learners (e.g. research papers with undergraduate students). As found, adults are not necessarily proficient estimators. As a matter of fact, high ability in exact calculations is not in alignment with their performance in numerical situations which require estimation (Lemaire, Arnaud & Lecacheur, 2004. Hanson & Hogan, 2000).
The principal idea for the design of the present study was to investigate the computational estimation ability of adult learners, since most research papers about estimation concern students or teachers. We strongly suggest that estimation is more useful for adults since they deal with money and transactions every day. This is why adults’ participation in an intervention related to estimation ability was another important objective of our research. Thereby, it can be examined if instruction is really beneficial for adult learners and if there are any specific boundaries or advantages that should be taken into account. Consequently, an adult secondary school appealed to us as an inspiring educational setting to implement our study.

2. THEORETICAL BACKGROUND

Mental calculations have been incorporated into the curriculum with the intention of introducing a number sense approach, since written computations had a long dominance in mathematics education before (McIntosh, 2004). Likewise, computational estimation has been included due to its practical utility for predicting or checking a result (Segovia & Castro, 2009) apart from its benefit for the development of number sense. Therefore, mental procedures can produce either an exact or an approximate arithmetic outcome depending on the situation and the purpose of the mental operation.

Computational estimation has been thoroughly investigated and many factors affecting computational estimation have been found. First of all, number type or operation type of problems can influence computational estimation skill. Tsao & Pan (2011) found that students perform better in estimation with natural numbers and worse in problems with fractions. Tsao (2013) documented elementary teachers’ low performance on estimations involving multiplication and division. These difficulties suggest that a deep understanding of rational numbers and multiplicative reasoning is necessary for estimation and imply a strong relationship between estimation and number sense, which includes knowledge of and facility with place value, numbers and operations.

Afterwards, cultural factors, such as national educational systems or social values about mathematics, can influence students’ performance and attitudes towards estimation (Liu & Neber, 2012). Imbo & LeFevre (2011) also reported differences between different ethnic groups: Chinese selected computational estimation strategies less adaptively than the Belgians, although they were faster and more accurate. The writers explained this finding in terms of previous educational experience and attitudes towards estimation (Asians prefer exactness).

Next, attitudes can influence the value credited to estimation. Tsao (2013) found that pre-service elementary teachers hold computational estimation in high regard. However, their attitudes were neutral in general, because they rarely applied estimation strategies and, as a result, had insufficient experience in computational estimation. Except for the teachers, some learners may also feel uneasy about
estimation, since answers in school mathematics are expected to be “right” or “wrong” and must be exact (Newmarch & Part, 2007).

Last but not least, age is crucial for the development of estimation skills. Longitudinal studies showed that computational estimation skill develops with age, given that it depends on learning number facts and problem-solving procedures, such as rounding (Booth & Siegler, 2006). Psychological research with executive functions aligns with previous studies demonstrating that strategy selection and execution in computational estimation tasks improve with age (Lemaire & Lecacheur, 2011). Finally, Lemaire, Arnaud and Lecacheur (2004) documented age-related strategic changes in computational estimation performance, as older adults provided less accurate estimates than younger ones, although both groups had similar strategy preferences.

Considering that adults can draw on their everyday experience and judgments (e.g. are 4€ enough to buy a bread and a carton of milk?), they should be expected to be fluent estimators. However, research does not report such facility with estimation. For instance, Hanson & Hogan (2000) found that adults’ estimation skill was much worse than their computational skill. Their preference to exact calculations can be probably interpreted as a result of limited prior experience in computational estimation in formal education.

It is surprising, though, that most research focuses on students or on teachers and only few recent studies about estimation concern adults. In fact, it could be claimed that estimation is more necessary, useful and beneficial for adults, because they deal with money and transactions all the time. In addition, adults are at advantage because they can easily utilize their daily activities in order to learn estimation strategies, while young students may have some trouble. Indeed, Yang & Wu (2012) documented students’ difficulty in contextual computational estimation problems, because they needed to transform verbal words into number symbols. On the other hand, adult learners can engage in contexts with time and money in order to learn strategies and see the value of estimation (Ness & Bouch, 2007).

On this account, computational estimation should be considered to be introduced in adult numeracy programmes. Lemonidis (2010) proposed the incorporation of estimation in the teaching practices of mathematicians in Greek Second Chance Schools, as it constitutes an integral part of number sense and operations. Second Chance Schools offer a two-year intensive programme for adults and issue their graduates with a leaving certificate equivalent to a Junior High School certificate, which is typically awarded upon the completion of compulsory education. Having completed primary education is the only requirement for the enrolment.

The aim of this study is to investigate adult learners’ computational estimation skills in a Greek Second Chance School and to implement a teaching experiment with a series of computational estimation activities. The present study focused on students’ performance and mental strategies with the intention to explore factors that can affect estimation skill. Additionally, the feasibility of teaching computational
estimation in such schools was tested with the scope of introducing mathematical topics that are meaningful, useful and beneficial for adult learners.

3. METHOD

Participants. Fifteen adults (10 males), aged from 19 to 65 (mean age 33 years and 3 months) participated. Four of them come from Albania. They all have a primary school leaving certificate and were attending the first year of a Second Chance School in North Greece during this study. One of them is a pensioner, six (all males) were working and the remaining eight were unemployed at that time.

The teacher. The teacher who engaged in the professional learning intervention and conducted the experimental teaching sessions is a mathematician with a M.Sc. in theoretical calculus. She worked as a supply teacher in secondary education for the last four years, although her studies did not include any subjects about mathematics education and neither had she a further training in teaching maths.

Preparing for the experiment. The researcher contacted the maths teacher and observed her lessons for five weeks. His intent was to meet the participants in their original learning ecology and to estimate their skills in mathematics in order to ensure that the experiment would correspond to their needs. That is why he noticed the tasks they were typically asked to solve and the materials they used, the kinds of classroom discourse and the norms of participation.

As observed, the instruction was mostly teacher-centred. The mathematician usually used school textbooks as educational resource and distributed photocopies with solved examples of numerical problems. The copies were also accompanied by a worksheet of tasks. The vast majority of the tasks included pure numerical problems. Adult learners were asked to solve them on the whiteboard and, when needed, she demonstrated the steps required for the solution. Most interactions focused more on the procedural and less on the conceptual understanding. No classroom technology tools were used.

Provided that the researchers did not intent to change the classroom teacher’s practices, they designed three worksheets with tasks. The design of the activities was consistent with Van den Heuvel-Panhuizen’s (2001) learning-teaching trajectory. The first worksheet was about rounding (e.g. [census figures are provided] how would you indicate the number of residents in an information booklet of our local prefecture?). The second included estimation problems with addition and subtraction (e.g. Jacob & Martha want to buy these products [...] –are 30€ enough?). The last worksheet was about estimation with multiplication and division (e.g. can you help Sophie note the decimal point in the following operation?). The designed activities were adjusted to the local educational setting. For example, fractions were excluded from the estimation tasks that were going to be used as material in the forthcoming experimental teaching sessions.
Two changes were adopted in the teaching practices. First, the estimation worksheets contained many contextual problems. This direction was in accordance with the nature of estimation and with the needs of an adult numeracy instruction. Second, adult learners were allowed to use calculators during the instruction.

Furthermore, the teacher participated in a two-hour professional learning intervention (organized by the researcher) about computational estimation and its teaching. Among other things, the concept of estimation, common strategies and its necessity in mathematics and in mathematics education were discussed. Eventually, she also familiarized with the activities designed for the teaching intervention.

Conducting the experiment – Data sources. Many methods were used to collect data: pre- and post-tests, questionnaires, observation and teacher’s interview.

Students’ computational ability was examined through individual interviews before and after the teaching intervention (pre- and post-test respectively). Interviews took place in school and lasted approximately 15 minutes each. For that reason, an instrument with 9 questions was developed by the researchers, adapted to the students’ needs. As a result, it included only one- or two-digit numbers. Four tasks regarded estimation in addition and subtraction (e.g. Harry bought an electric shaver. Its initial price was 69€ but he bought it for 37€. The discount was over or under 40€? Explain.). The other five tasks included numerical situations which required estimation in multiplication or division (e.g. Zoe, a primary teacher, bought 67 candies for her 10 students. Were there enough candies to take 7 each? Explain.). Each pre- and post-test task had the same numerical data but the context slightly changed.

A questionnaire with 12 items on a four-point Likert-type scale (-2=disagree a lot, ..., 2=agree a lot) was used to measure students’ attitudes toward mathematics. Reverse scoring was also used for statements in negative terms. The items focused on three aspects of attitudes toward mathematics:

1. confident (e.g. I am good at mathematics),
2. like learning (e.g. mathematics is boring),
3. value (e.g. mathematics is not useful for most occupations).

The categories were in alignment with TIMSS 2011 context questionnaire scales (Martin & Mullis, 2012) but were slightly adapted for adults.

The teaching experiment lasted for three weeks (7 lessons particularly). Its conduction began a week after pre-test and attitudes measurement. The first author was present during the teaching sessions. A witness is necessary, according to Steffe & Thompson (2000). Field notes were recorded through observation.

A semi-structured interview was conducted with the maths teacher after the teaching intervention with the view to exploring her ideas and beliefs about computational estimation and her teaching experience.
4. RESULTS

4.1. Performance on the 9-task tests (N=13)

Students’ responses were classified into categories as shown in Table 1. Mental responses, either given by an exact mental calculation or by a computational estimation, represent the majority of their answers (58.1% and 82.1% in the pre- and post-test respectively). It is obvious that most mental responses resulted from computational estimation strategies. Responses based on written algorithms or on pocket calculator were less common in both tests. Additionally, there were some correct responses that could not be sufficiently explained by the participants (categorizes as “vague explanation”). Finally, about one fifth of the responses in the pre-test were wrong. However, the mean score of the participants’ errors was almost reduced to the half in the post-test.

<table>
<thead>
<tr>
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<th>pre-test Frequency</th>
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<th>post-test Frequency</th>
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<td>60.7</td>
</tr>
<tr>
<td>Exact mental calculation</td>
<td>27</td>
<td>25.0</td>
<td>25</td>
<td>21.4</td>
</tr>
<tr>
<td>Written algorithm</td>
<td>9</td>
<td>7.7</td>
<td>2</td>
<td>1.6</td>
</tr>
<tr>
<td>Vague explanation</td>
<td>6</td>
<td>5.1</td>
<td>3</td>
<td>2.6</td>
</tr>
<tr>
<td>Pocket calculator</td>
<td>11</td>
<td>9.4</td>
<td>3</td>
<td>2.6</td>
</tr>
<tr>
<td>Wrong answer</td>
<td>23</td>
<td>19.7</td>
<td>13</td>
<td>11.1</td>
</tr>
</tbody>
</table>

According to the preceding data, it can be assumed from the pre-test that the adult learners are capable of doing mental calculations and computational estimations, although they had not previously received an explicit instruction in this field. Afterwards, the results from the post-test suggest that the students’ mental ability was significantly improved after the teaching intervention, since mental responses increased and all the other scores were reduced.

As shown in Table 2, a performance by type of operation analysis showed that estimation in addition or subtraction was easier than estimating in multiplication or division in both tests. Additionally, the mean score of mental responses increased in both categories and the gap closed. This implies that the teaching of estimation led to a deeper understanding of operations.

<table>
<thead>
<tr>
<th>Operation type</th>
<th>pre-test</th>
<th>post-test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition-subtraction</td>
<td>65.4 % (2.62 out of 4 tasks)</td>
<td>86.5 % (3.46 out of 4 tasks)</td>
</tr>
<tr>
<td>Multiplication-division</td>
<td>52.3 % (2.62 out of 5 tasks)</td>
<td>78.5 % (3.92 out of 5 tasks)</td>
</tr>
</tbody>
</table>
Next, the question is whether an improvement was noticed for all participants. As a matter of fact, the individual performance of every student was improved due to the teaching experiment. Participants’ mental responses per test are presented in Figure 1, in which pre-test mental performance scores are presented in descending order with S1 having the best mental performance and S13 the worst.

**Figure 1:** Frequency of mental responses by student per test (max=9)

As shown above, the 7 lessons about computational estimation were really beneficial for the thirteen students, since each had a better score in the post-test. Furthermore, their improvement was evident not only in their performance, but also in their wrong answers and their mental strategies as well.

Regarding the wrong responses, they were reduced from 23 (19.7% of total answers) to 13 (11.1%). Most incorrect answers were given at random or could not be sufficiently explained. The error types are presented in Table 3. It can be strongly suggested that the teaching experiment eliminated students’ errors.

**Table 3:** Frequency of error types per test

<table>
<thead>
<tr>
<th>Type of error</th>
<th>pre-test Frequency</th>
<th>Mean (%)</th>
<th>post-test Frequency</th>
<th>Mean (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wrong operation</td>
<td>1</td>
<td>0.9</td>
<td>1</td>
<td>0.9</td>
</tr>
<tr>
<td>Error in calculation</td>
<td>9</td>
<td>7.7</td>
<td>1</td>
<td>0.9</td>
</tr>
<tr>
<td>Irrational explanation</td>
<td>6</td>
<td>5.1</td>
<td>4</td>
<td>3.3</td>
</tr>
<tr>
<td>Random answer</td>
<td>7</td>
<td>6.0</td>
<td>7</td>
<td>6.0</td>
</tr>
</tbody>
</table>

As already demonstrated above, it has been found that the correct responses with vague explanation decreased from 6 to 3 (see Table 1) and that wrong answers with irrational explanation were reduced from 6 to 4 (see Table 3). It can be assumed that a long-term practice with mental calculations and computational estimations could enhance number sense and improve metacognitive skills in the long run, similarly to the metacognitive improvement noticed in an experimental design research conducted by Bobis (1991).
An analysis of errors by type of operation showed that the participants had difficulties in estimation involving multiplication or division (see Table 4). The mean scores of errors decreased for all operations after the implementation of the teaching sessions. The improvement was bigger in estimation problems of multiplicative reasoning.

Table 4: Mean score of errors by operation type per test

<table>
<thead>
<tr>
<th>Operation type</th>
<th>pre-test</th>
<th>post-test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition-subtraction</td>
<td>11.5 %</td>
<td>7.7 % (0.46 out of 4 tasks)</td>
</tr>
<tr>
<td>Multiplication-division</td>
<td>26.1 %</td>
<td>13.9 % (1.31 out of 5 tasks)</td>
</tr>
</tbody>
</table>

Concerning students’ mental responses, their mental strategies are presented in Table 5. Exact mental calculations were frequently used in tasks with integers, while computational estimation strategies were mostly used in tasks with decimals. Rounding was the most common computational estimation strategy (21 times in pre-test, 50 in post-test). Front-end strategy was also frequent. Ultimately, other strategies were observed too.

Table 5: Frequency of strategies per test

<table>
<thead>
<tr>
<th>Mental strategy</th>
<th>pre-test</th>
<th>post-test</th>
<th>difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact mental calculation</td>
<td>30</td>
<td>26</td>
<td>-4</td>
</tr>
<tr>
<td>Rounding</td>
<td>17</td>
<td>43</td>
<td>+26</td>
</tr>
<tr>
<td>Front-end strategy &amp; compensation</td>
<td>6</td>
<td>13</td>
<td>+7</td>
</tr>
<tr>
<td>Rounding &amp; compensation</td>
<td>4</td>
<td>7</td>
<td>+3</td>
</tr>
<tr>
<td>Front-end strategy</td>
<td>8</td>
<td>3</td>
<td>-5</td>
</tr>
<tr>
<td>Substitution</td>
<td>3</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>Averaging</td>
<td>-</td>
<td>1</td>
<td>+1</td>
</tr>
<tr>
<td>Compatible Numbers</td>
<td>-</td>
<td>1</td>
<td>+1</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>68</strong></td>
<td><strong>96</strong></td>
<td><strong>+28</strong></td>
</tr>
</tbody>
</table>

As a result of the teaching intervention, a significant increase of mental strategies was found. Particularly, the frequency of rounding strategy was more than doubled. Compensation strategy use was also doubled in post-test. It is possible that this increase is due to teacher’s tendency to insist on the rounding rule and to ask for accuracy. Nevertheless, she utilized situation based rounding as well, in order to highlight the importance of the word problems’ context, though less frequently.

As can be seen, averaging and compatible numbers strategies were noticed incidentally in post-test. Despite the fact that the tests included tasks that facilitated the use of these strategies, the latter were not recorded more than once. This is because of the maths teacher’s omission to refer to any of these strategies in class, although she was asked by the researchers to teach them explicitly.
4.2. Performance of students with difficulties in math

Two participants were examined with 6-task tests that were easier than the tests used for the rest 13 students. Many reasons contributed to this decision. First, their difficulties in learning were observed by the first author during the preparation of the study. Second, the maths teacher insisted that an easier instrument should be used for their interviews. Third, they were both reluctant to participate in the study because of their low confidence. Finally, the psychologist of the school confirmed that they both had low confidence and difficulties in learning but stated that they had not been checked for specific learning disorders, though this possibility cannot be rejected.

As shown in Table 6, the first student (S14) preferred to give random answers and to use the pocket calculator. In spite of its use, her responses did not make any sense, since she repeated the same wrong patterns. For instance, in task 6 she could not calculate how much should each of five persons pay if the total debt was 51.34€. In fact, her random answers were much bigger than the total amount in both tests. The other student (S15), who did give a right answer to the easier tasks, was unwilling to use the pocket calculator. On the contrary, she responded at random.

Table 6: Students’ answers by task per test

<table>
<thead>
<tr>
<th>Task</th>
<th>pre-test</th>
<th>post-test</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S14</td>
<td>S15</td>
</tr>
<tr>
<td>Task 1</td>
<td>Random answer</td>
<td>✓ Front-end strategy</td>
</tr>
<tr>
<td>4.75+4.65</td>
<td>&gt; or &lt; than 10?</td>
<td>2. Pocket calculator: 4+75=79 and 4+65=69</td>
</tr>
<tr>
<td>Task 2</td>
<td>1. Random answer</td>
<td>✓ Exact mental calculation</td>
</tr>
<tr>
<td>2 + 7 + 4 +</td>
<td>2. Pocket calculator: 2+7=9, 4+6=10, 3+6=9, could not decide if it’s equal to 9 or 10</td>
<td></td>
</tr>
<tr>
<td>6 + 3 + 6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 3</td>
<td>Random answer</td>
<td>Random answer</td>
</tr>
<tr>
<td>7.05+6.89</td>
<td>+7.12+6.75</td>
<td>+7.35</td>
</tr>
<tr>
<td>Task 4</td>
<td>1. Random answer</td>
<td>✓ Around 2, could not explain why.</td>
</tr>
<tr>
<td>9 : 5</td>
<td>2. Pocket calculator: 1+1=2, 1+1=2, 1+1=2. So it’s 6.</td>
<td></td>
</tr>
<tr>
<td>Task 5</td>
<td>Random answer</td>
<td>Random answer</td>
</tr>
<tr>
<td>3 x 28</td>
<td></td>
<td>28 x 3 = 84.</td>
</tr>
<tr>
<td>Task 6</td>
<td>Pocket calculator: 255 (wrong operation)</td>
<td>Random answer</td>
</tr>
<tr>
<td>51.34 : 5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### 4.3. Factors affecting computational estimation skill

**Prior Knowledge & Experience**

The importance of prior knowledge and experience was prominent in this study. Actually, even the data sources varied in accordance to the cognitive level of the participants provided that two different instruments were used depending on the target sample. Obviously, the students with difficulties in mathematics had not memorized as many arithmetic facts as the rest participants. S14 needed a calculator even for one-digit numbers addition. Additionally, it can be claimed that they did not possess variety of procedures for solving the estimation problems. As a matter of fact, they did not even cultivate their rounding skill after the teaching intervention, since there was no significant improvement in most cases. It is to be questioned, however, whether their bad performance is a result of low number sense or whether it is due to mathematical anxiety.

Afterwards, as shown in Tables 2 & 4, the type of operation had a strong effect on participants’ mental performance, since estimating products or quotients was found to be more difficult than estimating sums or differences. As can be seen in Figure 2, in both tests the adult learners demonstrated higher mean score on estimation tasks with addition or subtraction in comparison to problems with multiplication or division.

![Figure 2: Mean (%) of mental responses by operation type per test](image)

This finding implies that estimation skill depends on computational skill or on number sense in general. This is why good estimators are expected to be fluent in operations or to have a good number sense. Such a hypothesis was tested in the post-test interviews.

Indeed, personal information about the four participants with the highest performance showed that the best estimators had a strong cognitive background. In fact, the best estimators engaged in everyday activities that required mental procedures, like calculating mentally or estimating measures. It is suggested that their prior knowledge and experience was critical for estimating.
First of all, a surprising relationship between computational and measurement estimation is implied. Both participants with the highest mental performance (S1 & S2) measure lengths and surfaces all the time at work. Particularly, their jobs involve measurements of plasterboards (S1) and of tiles (S2). Consequently, they are both considered to be good at measurement estimation, since fluency in measuring is apparently necessary for installing plasterboards or floorings.

Next, except for experience in measurement, frequent use of mental calculations involving proportional reasoning is also suggested to be connected with high performance on computational estimation. The other two good estimators (S3 & S4), who demonstrated very high performance in both tests, do mental calculations of proportions frequently.

Student 3 (S3), who works at a bakery, claimed that he calculates the quantities of the ingredients depending on the number of loaves he wants to bake. As he said during his interview: “For example, at the bakery we put 33g of salt in 15L of water. If you have 16L or 10L you have to calculate how much [salt] you need –and you have to do it with your mind at that moment! Also, 1L of water requires 2kg flour and makes 6 loaves of bread. Depending on the bread I want, I calculate the flour.”

Likewise, student 4 (S4) also does mental calculations in her everyday life. In fact, she had been an immigrant in Germany for many years, before she came back to Greece. As a result, she had used both Drachma (Greek former currency) and Deutsche Mark (German former currency) before the transition to Euros, which is the actual currency for both countries. Thus, S4 used Drachma as a benchmark in order to link the other currencies to it. As she said: “I calculated everything with my mind. I did "drachmaization"... both with Marks and with Euros. My husband found it difficult but I could instantly calculate everything in Drachmas.”

To sum up, dealing with everyday activities that involve mental procedures like calculating proportions mentally or estimating measures was reflected as a strong advantage in the performance of the best estimators. Still, it seems possible that they have a positive attitude when they carry through these meaningful activities.

Unlike the best estimators, S13 (with the worst performance) did not reflect his prior experience with money in the pre-test. S13 is a delivery boy and he has to calculate mentally the change he has to give when he delivers the orders. However, as he confessed during the post-test interview, he does not enjoy mental calculations and he prefers that his boss calculates the change for him, before he drives off. For that reason, it is suggested that a positive attitude is necessary to benefit from cognitively demanding everyday activities.

Finally, the rest eight participants (S5 to S12) did not report any frequent or professional involvement in everyday activities that require calculations or measurements.

**Attitudes towards mathematics**

The mean scores of students’ attitudes toward mathematics are presented in Table 7. As shown, most students feel confident, like learning and give high value to
mathematics in general (mean scores > 1). A clear correlation between these attitudes and computational estimation skill cannot be inferred.

**Table 7:** Mean scores and standard deviations in students’ responses concerning their attitudes toward mathematics (confident, like learning and value)

<table>
<thead>
<tr>
<th>Student</th>
<th>Confident Mean (min=−2. max= 2)</th>
<th>S.D.</th>
<th>Like Learning Mean (min=−2. max= 2)</th>
<th>S.D.</th>
<th>Value Mean (min=−2. max= 2)</th>
<th>S.D.</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>2.00</td>
<td>0.00</td>
<td>1.75</td>
<td>0.50</td>
<td>2.00</td>
<td>0.00</td>
</tr>
<tr>
<td>S2</td>
<td>2.00</td>
<td>0.00</td>
<td>1.75</td>
<td>0.50</td>
<td>2.00</td>
<td>0.00</td>
</tr>
<tr>
<td>S3</td>
<td>1.00</td>
<td>1.41</td>
<td>1.75</td>
<td>0.50</td>
<td>1.75</td>
<td>0.50</td>
</tr>
<tr>
<td>S4</td>
<td>1.00</td>
<td>1.41</td>
<td>1.25</td>
<td>1.50</td>
<td>1.75</td>
<td>0.50</td>
</tr>
<tr>
<td>S5</td>
<td>2.00</td>
<td>0.00</td>
<td>2.00</td>
<td>0.00</td>
<td>1.75</td>
<td>0.50</td>
</tr>
<tr>
<td>S6</td>
<td>0.25</td>
<td>1.50</td>
<td>1.00</td>
<td>2.00</td>
<td>-0.25</td>
<td>2.06</td>
</tr>
<tr>
<td>S7</td>
<td>1.00</td>
<td>1.41</td>
<td>2.00</td>
<td>0.00</td>
<td>2.00</td>
<td>0.00</td>
</tr>
<tr>
<td>S8</td>
<td>1.75</td>
<td>0.50</td>
<td>2.00</td>
<td>0.00</td>
<td>1.75</td>
<td>0.50</td>
</tr>
<tr>
<td>S9</td>
<td>-0.25</td>
<td>1.50</td>
<td>1.75</td>
<td>0.50</td>
<td>1.25</td>
<td>1.50</td>
</tr>
<tr>
<td>S10</td>
<td>0.50</td>
<td>1.00</td>
<td>1.75</td>
<td>0.50</td>
<td>1.00</td>
<td>2.00</td>
</tr>
<tr>
<td>S11</td>
<td>1.25</td>
<td>0.50</td>
<td>1.50</td>
<td>0.58</td>
<td>1.75</td>
<td>0.50</td>
</tr>
<tr>
<td>S12</td>
<td>2.00</td>
<td>0.00</td>
<td>2.00</td>
<td>0.00</td>
<td>1.00</td>
<td>2.00</td>
</tr>
<tr>
<td>S13</td>
<td>1.50</td>
<td>0.58</td>
<td>2.00</td>
<td>0.00</td>
<td>0.75</td>
<td>1.89</td>
</tr>
<tr>
<td>S14</td>
<td>-0.75</td>
<td>1.89</td>
<td>0.50</td>
<td>1.73</td>
<td>-0.75</td>
<td>1.89</td>
</tr>
<tr>
<td>S15</td>
<td>0.25</td>
<td>1.50</td>
<td>1.75</td>
<td>0.50</td>
<td>1.25</td>
<td>1.50</td>
</tr>
<tr>
<td>Mean (N=15)</td>
<td>1.03</td>
<td>1.65</td>
<td>1.27</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It is evident, however, that the participants with difficulties in mathematics (S14 & S15), who underperformed, have low confidence in mathematics (-0.75 and 0.25 respectively). It can be argued that it is about a correlation between low confidence and low performance and not an inferential relation, because the cause of their behaviour is probably much more complex. For instance, they may suffer from mathematical anxiety or they could have learning difficulties.

**4.4. Maths teacher’s beliefs about computational estimation**

The classroom maths teacher was interviewed after she implemented the teaching experiment. She said that she was unaware of the concept of computational estimation before her professional learning intervention which was conducted during this study. The only relevant knowledge she had was the rounding strategy, which is instructed in junior high school.

However, it is impressive that she could discriminate rounding from computational estimation after the teaching intervention. It can be assumed that she has a good conceptual understanding of computational estimation due to the experience she gained in this study. First, she understands that estimation reflects
individual approaches. As she said: “in rounding I did not let students free... when I asked them to "round off a number" I told them to which position they should do it. [...] in estimation they can choose [the position].” Second, she claimed that the fairest direction to round off “depends on the problem”. Third, she mentioned the “different procedures” with which estimation problems could be solved implying that rounding is not the only estimation strategy. For instance, she said that “sometimes we only take the first digit under consideration” having the front-end strategy in mind.

Although the maths teacher displayed conceptual understanding of estimation in her interview, she did not demonstrate the expected procedural knowledge in the classroom. Lack of such knowledge of computational estimation strategies can hardly be called into question. Thereby, it is suggested that she has a conservative stance about estimation. Actually, she avoided instructing various computational estimation strategies during the experimental teaching, whilst she overemphasized the rounding off procedure. Therefore, this emphasis explains the significant increase of rounding strategy in post-tests.

The mathematician’s behavior may be attributed to an implicit belief that rounding is the only mathematically correct procedure to estimate, since it is based on rule. Many educators equate computational estimation with rounding, as Alajmi (2009) found. Additionally, as has been noted, estimation can be considered inferior to exact calculation by estimators who have a vague notion of its nature and purpose (Morgan, 1988, as cited in Sowder, 1992, p. 375). Maybe the maths teacher does not fully understand the meaning of estimation.

The truth is that the maths teacher was pretty confident about her content knowledge, because she believes that instrumental strategies are known to mathematicians. It is that she had not taught estimations before and she “only needed a direction” as she said. Overall, she referred to her teaching intervention as a “very good” experience. Furthermore, she characterized the concept of computational estimation as “easy” and “useful” topic, especially for those who are not fluent in the basic operations yet. Also, she found the context of the word problems intriguing and she commented their link with everyday life as something positive.

Although she believes that students’ performance was improved due to their long encounter with estimation (the 7-hour intervention allowed a deep understanding), she commented that she devoted too much time. Namely, she made clear that she felt stretched to teach other mathematical topics too. Maybe this is why her comments were moderate and she was not enthusiastic during the interview. Finally, there was not any spontaneous disclosure of intention to teach estimation to her future students. Thus, it is doubtful that she would engage estimation in her teaching practices.

5. DISCUSSION

The results of the present study documented adults’ computational estimation skills. However, some limitation of the research cannot be ignored. The sample was small,
only a class of 15 adult learners participated. In addition, the tests for the interviews and the materials used for the teaching sessions were adjusted to the specific participants and to the original learning ecology. Due to the small sample and to the nature of the present investigative study, it was not possible to conduct further statistical data analysis that could provide a bit more generalizable results. Besides, the use of typical quantitative assessment tests and methods was avoided, because the methodological choices served the qualitative objectives of the current study. This is why many aspects about the teaching and learning of estimation in Second Chance Schools were revealed.

Regarding participants’ estimation skills, most responses in pre-test (58.1% of total answers) derived from mental procedures, like computational estimation strategies or mental calculations. Rounding was the most frequent estimation strategy, a result that is in accordance with other researches with various samples (Hanson & Hogan, 2000. Tsao & Pan, 2013. Boz & Bulut, 2012. Alajmi, 2009). However, although adults can estimate, there is still much room for improvement.

The teaching experiment conducted in this study investigated some factors that can be crucial for estimation. First, prior knowledge and experience seems to be fundamental for estimation skill. On the one hand, the best estimators had a strong cognitive background deriving from their everyday activities, which involved mental procedures like measuring or calculation proportions. Their everyday experience was reflected in their estimation performance. Lemonidis (2013) suggests that craftsmen like builders are fluent in computational estimations, because they are good at measurements. On the other hand, students with difficulties in maths did not even demonstrate a slight improvement in their rounding skill. This means that they even lack the knowledge of basic procedures.

Next, participants committed most errors in multiplicative reasoning estimation tasks and had a better performance in estimation involving addition and subtraction. Facility with additive estimation problems is also reported by other researchers (Tsao, 2013. Bana & Dolma, 2004. Hanson & Hogan, 2000). Thus, teachers need to evaluate students’ skills in mathematics and help them enhance their number sense by promoting a deeper understanding of numbers and operations. Especially for adults, teachers can use their everyday activities with money as a source of emotional engagement in order to motivate them to improve and practice their estimation skills.

In accordance with the results above, Star et al. (2009) also exhibited the role of prior knowledge in the development of estimation strategy flexibility. They found that students who were fluent estimators at pre-test used to increase their accuracy in estimation, while less fluent students adopted strategies that were easier to implement. In another study, Star & Rittle-Johnson (2009) showed that students who already knew some estimation strategies gained a deeper conceptual understanding of estimation after getting taught some lessons about estimation.

Afterwards, attitudes towards estimation are also important. Participants had a positive attitude towards mathematics in general, except for the two women (S14 &
S15) with the difficulties in maths who were not confident at all. However, their topic-specific attitudes were not investigated. Maybe some students, like S13, did not enjoy mental calculations in particular or did not consider approximate outcomes mathematically correct. Even the maths teacher avoided teaching a mixture of estimation strategies and insisted on rule-based rounding, although she recognized the usefulness of estimation for weak students and she seemed to understand the concept of estimation. Neutral attitudes like this were also reported by Tsao (2013) who explained such elementary teachers’ stances as a result of limited school experience in estimation. This is why estimation should be included in teacher training education programmes too.

How can maths educators improve such negative or neutral attitudes towards estimation? By creating positive experiences for both students and teachers. Professional learning interventions have been designed in order to develop teacher’s PCK about computational estimation and to enhance students’ performance in estimation (Mildenhall & Hackling, 2012. Mildenhall et al., 2009). Likewise, in this study, the maths teacher was involved in a professional learning intervention and implemented 7 teaching sessions about estimation. Consequently, the introduction of this topic broadened her teaching practices and was also beneficial for adult learners. Particularly, participants’ mental strategies covered 58.1% of total answers in pre-test and 82.1% in post-test. Their improved performance was also accompanied by a decrease of errors as well.

For all the reasons mentioned before, it is strongly suggested that computational estimation is incorporated into Second Chance Schools and into adult numeracy teaching practices in general. Adult learners will improve their performance and their strategies in estimation. In addition, they are expected to be motivated by contextual estimation problems, because they regard their daily activities. Finally, maths teachers are advised to make sure that all students understand the meaning of estimation and improve their procedural knowledge by enhancing their number sense. In this fashion, even the students who are at disadvantage are going to be helped.

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**BRIEF BIOGRAPHIES**

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STUDENTS’ BEHAVIORS IN COMPUTATIONAL ESTIMATION CORRELATED WITH THEIR PROBLEM-SOLVING ABILITY

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ABSTRACT
The present study examined the behaviors in number sense and especially in computational estimation problems, of 5th and 6th grade students of Greek Primary schools, who participated in a mathematical competition. These students have not received any training in relation to computational estimation strategies nor have they been trained in expressing themselves verbally or in writing as to their way of thought while performing the set tasks. Therefore, it was examined whether these students know how to perform computational estimation, which strategies they use, and what errors they make. It was also examined whether they can write down their explanations in computational estimation problems and this ability was compared with their problem solving ability. Many researchers (Buchanan 1978; Reys 1986; Reys and Reys 1989; Dowker 2003) state that estimation plays an important role in problem solving and is very similar to it, however, no experiments have been conducted specifically on this issue.

Keywords: Number sense, computational estimation, strategies of computational estimation, problem solving, metacognition.

1. INTRODUCTION
For many years, computational estimation has been identified as a basic skill in mathematics education (e.g. NCTM 1980, 1989, 2000). In Greece, it has only recently (2003) been included in school curricula (Δ.Ε.Π.Π.Σ.1 – Α.Π.Σ. 2003) and was applied in classrooms in 2006 when new school textbooks were introduced. Nevertheless, apart from reference in the program and some exercises in the school textbooks, there has not been any significant evidence of teaching computational estimation through the use of different strategies and students’ practice.

1 A cross thematic curriculum framework for compulsory education
Computational estimation is considered to be a main element of number sense (Greeno 1991; McIntosh 2004). Sowder (1984) claims that people who are good at estimation have a good understanding of basic facts, position value and arithmetic properties. They are also capable of performing mental calculations, are tolerant to error, and flexible enough to use a variety of strategies while they also have high self-esteem.

Estimation plays an important role in problem solving (Buchanan 1978; Reys and Reys 1989; Dowker 2003). Reys (1986) claims that estimation is very similar to problem solving and requires a variety of skills and attitudes towards the specific subject, while it can be developed and improved over time. Someone who is good at estimating chooses a strategy in accordance to the problem, along with special numbers and operations. Estimation, just as problem solving, is an issue which cannot be isolated or/and taught separately. It is related through many different curriculum subject fields, and in order to be effectively developed it should be cultivated through long-term procedures during mathematics studying and teaching.

Many studies show that students face difficulties in estimation (Levine 1982; Sowder & Wheeler 1989; Reysetal 1991a; Hanson & Hogan 2000; Lemaire & Lecacheur 2002; Siegler & Booth 2004; Mildenhall et al. 2009). Researchers believe that these difficulties are connected with their deficiency to realize the meaning of the estimation procedure. Sowder and Wheeler (1989) suggest that students make errors which are usually of a conceptual nature. Some of these errors show that students do not fully comprehend what estimation is whereas some other errors are the outcome of their inability to understand the procedure through which one can perform estimation. Adults display similar behavior, for example, many prospective teachers do not know the way to perform computational estimation in given problems, and they perform accurate calculation using written algorithms (Lemonidis & Kaimakami 2013).

Computational estimation strategies have been researched concerning different levels of calculation ability of adults (Dowker 1992; Dowker et al. 1996), children and teenagers (Levine 1982; Baroody 1989; Sowder & Wheeler 1989; Reys et al. 1991a; Dowker 1997; Lemaire et al. 2000), and target groups of different ages (LeFevre et al. 1993; Lemaire & Lecacheur 2002). The different computational estimation strategies can be categorized according to different levels of generalization. Generally, it has been found that children and adults make use of the following three groups of strategies: reformulation, compensation and translation (Reys et al. 1982; Reys et al. 1991b; Sowder& Wheeler 1989).

Levine (1982) interviewed college students of varying mathematical backgrounds to identify strategies used to estimate. Each student was asked to estimate the answers to 20 questions. Levine (1982) found that poor estimators preferred to look for the exact computation, and then rounded to find an estimate. She states that this method did not “require the individual to sense any relationships or to have any 'number sense' to carry it out” (p. 358). Good estimators in Levine's study used more
strategies and appeared to be more flexible in their thinking than poor estimators (Levine, 1982). Yang (2005) in an interview setting, asked a series of questions to 21 6th-graders from four public schools in south Taiwan, related to whole and decimal numbers, designed to assess their number sense. The results showed that rule-based methods, or no explanation, were the most popular responses for the students who answered correctly at each level—e.g. low group (11 out of 15), middle group (22 out of 29) and high group (18 out of 26) (p.321).

The outcome of these research shows that quite early in the computational estimation developmental trajectory, children use a variety of strategies. These strategies are often naturally dependent on the estimation operations given. Furthermore, there are different names given by different authors. According to these research we can mention the following strategies: rounding and compensation, front – end strategy, special numbers strategy, and proceeding algorithmically (see examples below).

Hall (1977) and Hallowell (1977) concluded that estimation ability is related to problem-solving ability. Paull (1972) found that estimation of numerical computation is significantly correlated with problem solving, mathematical ability, and verbal ability, and the ability to compute rapidly was related to the ability to estimate numerical computation. Wai and Kheong (1998) examined the correlation between problem-solving ability, mental calculation ability and estimation ability of primary 5th grade pupils. 567 pupils from 4 Singapore schools were administered a problem-solving, a mental calculation and an estimation test. Results from quantitative analyses showed that the scores of problem solving, mental calculation and estimation were significantly correlated. The correlation between problem-solving and estimation was 0.61.

2. THE PRESENT STUDY

2.1. Research questions

The sample in the case considered involved selected students of two final grades of primary school (5th and 6th), who were not taught computational estimation. The research questions were the following:

- How do these students deal with computational estimation problems and which strategies do they use?
- What are the common errors they make and what are their weaknesses in solving computational estimation problems?
- Can these students present their way of solving computational estimation problems in writing?
- Is the ability to solve computational estimation problems associated with metacognitive abilities as well as problem solving ones?
2.2. Methodology

The sample
596 students of 5th and 6th grade of primary school participated in this study. The students took part in the 8th “Nature and Life Mathematics” competition in which schools of Western Macedonia and Serres participated. 314 of them were fifth grade students, 282 were sixth grade students. In relation to gender, 304 (51%) were boys and 292 (49%) were girls. Students who took part in this competition were not selected. Their participation in the competition was completely voluntary. Their positive attitudes towards mathematics were perhaps their most distinctive feature.

Examination Procedure
This competition was held in May 2012 and took place in a time slot beyond the school timetable. It was a written examination and the computational estimation problems were one subject out of the four given. The other three subjects were word problems. In each computational estimation problem, students were required to write down their way of thinking, their own strategy for solving the problem.

The Problems
The computational estimation problems were the following:

5th Grade
Solve the above problems by mental computation and without using written operations. Explain the way you thought.
1.P.5) Mary ran 1/2 km in the morning and 3/8 km in the afternoon. Did she run at least 1 km?
2.P.5) A worker worked for 28 days earning 56 € per day. How much money did he approximately earn?

6th Grade
Solve the above problems by mental computation and without using written operations. Explain the way you thought.
1.P.6) In 816 ml of a substance 9.84% is alcohol. How much alcohol is approximately in the substance?
2.P.6) Give an approximate estimate of the sum of the following amounts of money:
1.26 €, 4.79 €, 0.99 €, 1.37 €, 2.58 €

We also present the three word problems for the fifth grade:
1) Kids’ animals. Nikos, Danai, Helen and John each have one of the animals: a cat, a dog, a parrot and a goldfish. Danai has an animal which doesn’t live under water. John has a four-leg animal. Helen has a bird and Danai hasn’t got a cat. What animal does each one have?
2) Birds on trees. There were 60 birds on three trees. Then 6 birds flew from the first tree, 8 birds from the second tree and 4 birds from the third tree. After
that, the number of the birds on each tree was the same. How many birds there were on the second tree at the beginning?

3) *The Square.* Helen cut a square paper with a perimeter of 20 cm into two rectangles. The perimeter of one of the rectangles is 16 cm. What is the perimeter of the second rectangle?

**The three word problems for the sixth grade:**

1) *Costas’ cat.* Costas’s cat drinks 60 ml milk when not chasing mice while drinking 80 ml milk when chasing mice. In 14 days he has chased mice one day on two. How much milk did he drink during the 14 days?

2) *Chocolates.* A box contains 14 chocolates, 8 in the form of snail, and the rest in the form of turtle. 7 chocolates are black and the rest is white. It contains exactly two turtles that are not black. How many white snails are there?

3) *Squares.* Below, Schemes I, II, III and IV are squares. The perimeter of the square I is 16 cm and the perimeter of the square II is 24 cm. What is the perimeter of the square IV;

![Squares Diagram](image)

**The strategies**

This study examines some appropriate computational estimation strategies used by Primary school graders to solve four problems.

The first problem for fifth graders (1.P.5.) and sixth graders (1.P.6.) required from students to estimate the sum of ½ and 3/8 and the percentage of 9.84% of 816 ml. To solve them, the most suitable strategy was the *special numbers* strategy. More specifically, in the first one the special number ½ where 3/8 < 1/2 and in the second one the special number 10%≈9.84% were in use. In problem 2.P.5., which posed the question of estimating the product 28x56, the most suitable strategy was *rounding with compensation:* 28x56≈30x50 or 30x60-100=1700. Finally, in the 2.P.6. problem for estimating the sum 1.26 + 4.79 + 0.99 + 1.37 + 2.58, the most appropriate strategy was the *front-end strategy:* 1+4+1+1+2 = 9 and 0.30 + 0.80 + 0.40 + 0.60 ≈2, 9+2=11.
2.3. Results

2.3.1. Students’ performance in computational estimation problems

Table 1: Students’ performance in computational estimation problems

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<thead>
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<tbody>
<tr>
<td></td>
<td>1/2 plus 3/8</td>
<td>28 x 56</td>
<td>9.84% of 816</td>
<td>1.26+4.79+0.99</td>
</tr>
<tr>
<td>Computational</td>
<td>89 (28.3%)</td>
<td>70 (22.3%)</td>
<td>97 (34.4%)</td>
<td>137 (48.6%)</td>
</tr>
<tr>
<td>estimation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exact calculation</td>
<td>123 (39.2%)</td>
<td>147 (46.8%)</td>
<td>20 (7.1%)</td>
<td>73 (26%)</td>
</tr>
<tr>
<td>with algorithm</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wrong answer</td>
<td>82 (26.1%)</td>
<td>93 (29.6%)</td>
<td>130 (46.1%)</td>
<td>48 (17%)</td>
</tr>
<tr>
<td>No answer</td>
<td>20 (6.4%)</td>
<td>4 (1.3%)</td>
<td>35 (12.4%)</td>
<td>24 (8.5%)</td>
</tr>
<tr>
<td>Total</td>
<td>314 (100%)</td>
<td>314 (100%)</td>
<td>282 (100%)</td>
<td>282 (100%)</td>
</tr>
</tbody>
</table>

In Table 1, the data show that a considerably small percentage of students were familiar with and could perform computational estimation problems. Concerning students of the 5th grade, in the 1st and the 2nd problems, 28.3% and 22.3% respectively were able to execute computational estimation correctly, while in the 6th grade the percentage was higher reaching 34.4% and 48.6% for each problem respectively. A considerably high percentage of the students attempted to solve problems employing exact calculation. In particular, 5th grade students used exact calculation more than computational estimation in solving problems. More specifically, the 1st and the 2nd problems were solved by means of employing exact calculation by 39.2% and 46.8% of the students. In the 6th grade, a small percentage of students (7.1%) solved accurately the first problem (1.P.6.) using exact calculation, as the written algorithm was hard to be used for solving it. Therefore, we came across a greater percentage of incorrect answers (46.1%) as well as failure to produce any answer (12.4%). Levine (1982), also found that poor estimators preferred to look for the exact computation, and then rounded to find an estimate.

2.3.2. Strategies used by students

In the following table (Table 2), the problem-solving strategies used by students who took part in the mathematics competition are presented.

Table 2: Percentages of strategies used in correct computational estimation answers

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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>½ plus 3/8</td>
<td>28 x 56</td>
<td>9.84% of 816</td>
<td>1.26+4.79+0.99+</td>
</tr>
<tr>
<td>Special numbers</td>
<td>1/2 is half and 3/8 is less</td>
<td>88 (28%)</td>
<td>59 (20.9%)</td>
<td></td>
<td>1.37+2.58</td>
</tr>
<tr>
<td></td>
<td>than the half strategy</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>9.84%=10%</td>
<td></td>
<td></td>
<td></td>
<td>(61%)</td>
</tr>
</tbody>
</table>
In the first problem (1.P.5.), less than 1/3 (28%) of the students used the special number strategy, the most suitable strategy for this exercise. The rest of them used exact calculation by summing the fractions.

In the second exercise (2.P.5.), among the few students who used rounding (22%), were students (16.6%) who performed rounding of both terms: 28x56=30x60, while some of them (5.4%) performed rounding and compensation, which was the correction of rounding.

In problem 1.P.6., only one-third of them (34.4%) managed to produce the correct computational estimation. Most of these students (20.9%) used the special numbers strategy; they interpreted 9.84% as 10%. The rest of the students (11.7%) used the rounding of 816=800 or 820 while simultaneously using the special number 9.84%=10%.

For almost half of the students (48.6%) who were able to perform a computational estimation in 2.P.6., most of them (38%) used rounding of addition terms (1.3+4.8+1+1.4+2.6=11), whereas very few students (7.8%) used the front-end strategy, and added separately the integer parts (1+4+1+1+2=9) and the decimal parts (0.30+0.80+0.40+0.60=2.10) of the numbers.

We can assume that the strategy known and used by the students is that of rounding (38% in 2.P.6. or 16.6% in 2.P.5.). Very few students can actually use number sense strategies or the most appropriate strategies for computational estimation, which are the special numbers strategy (28% in 1.P.5. and 20.9% in 1.P.6), the
rounding and compensation strategy (5.4% in 2.P.5. and 1% in 2.P.6.) and the front-end strategy (7.8% in 2.P.6). Finally, we can say that students don’t know well the number sense strategies for estimation. Similarly, Yang (2005), finds that rule-based methods, or no explanation, were the most popular responses for the 6th grade students.

2.3.3. Errors

<table>
<thead>
<tr>
<th>Type of error</th>
<th>1.P.5. 1/2plus 3/8</th>
<th>2.P.5. 28 x 56</th>
<th>1.P.6. 9.84% of 816</th>
<th>2.P.6. 1.26+4.79+0.99+1.37+2.58</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wrong application of written algorithm</td>
<td>38(12.1%) (46.34%)³</td>
<td>65(20.7%) (69.9%)</td>
<td>51(18%) (39.23%)*</td>
<td>17(6%) (35.4%)*</td>
</tr>
<tr>
<td>Wrong computational estimation</td>
<td>13(4.1%) (13.98%)*</td>
<td>18(6.4%) (13.85%)*</td>
<td>20(7.1%) (41.6%)*</td>
<td></td>
</tr>
<tr>
<td>Other operation</td>
<td>5(1.6%)</td>
<td>1(0.3%)</td>
<td>27(9.6%)</td>
<td>(20.77%)*</td>
</tr>
<tr>
<td>Solution without logical explanation</td>
<td>8(2.5%)</td>
<td>8(2.5%)</td>
<td>5(1.8%)</td>
<td>5(1.8%)</td>
</tr>
</tbody>
</table>

Students’ errors can be categorized in two groups: the wrong application of written algorithm and the production of incorrect computational estimation. Most written algorithm errors (20.7%) appeared during the performance of the multiplication 28x56 in 2.P.5., because this multiplication had four partial products and it was difficult to be calculated mentally. A lot of errors (18%) were also identified in 1.P.6. when students were trying to perform the multiplication 816x9.84 without rounding the numbers.

The error in 2.P.6. is a typical computational estimation error. 20 students (7.1%) gave answers such as: 1.30€ - 5.00€ - 1.00€ - 1.40€ - 2.60€, which means that they rounded each number and they did not add up in the end.

In 1.P.6., 16 students (5.7%) gave 10% as an answer. They therefore rounded 9.84 and provided it as the final solution number. For these errors we can infer that these students didn’t comprehend what computational estimation is. This finding agrees with other results from previous research where we can find that ‘Poor estimators have only a vague notion of the nature and purpose of estimation; they believe it to be inferior to exact calculation (Morgan, 1988) and equate it with guessing (Sowder, 1984)’ in Sowder, 1992, p. 375.

³ * The percentage which corresponds to total number of students who gave incorrect answers is marked in bold.
2.3.4. Students’ metacognitive ability in computational estimations

Table 4: Students’ percentages ability of their written problem-solving justifications

<table>
<thead>
<tr>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>½ plus 3/8</td>
<td>103 (32.8%)</td>
<td>161 (51.3%)</td>
<td>83 (29.4%)</td>
<td>115 (40.8%)</td>
</tr>
<tr>
<td>28 x 56</td>
<td>148 (47.1%)</td>
<td>106 (33.8%)</td>
<td>90 (31.9%)</td>
<td>113 (40.1%)</td>
</tr>
<tr>
<td>9.84% of 816</td>
<td>81 (25.8%)</td>
<td>59 (18.8%)</td>
<td>80 (28.4%)</td>
<td>94 (33.3%)</td>
</tr>
<tr>
<td>1.26+4.79+0.99+1.37+2.58</td>
<td>(91%)**</td>
<td>(84.3%)**</td>
<td>(82.47%)**</td>
<td>(68.6%)**</td>
</tr>
<tr>
<td>No explanation</td>
<td>20 (6.4%)</td>
<td>4 (1.3%)</td>
<td>30 (10.6%)</td>
<td>24 (8.5%)</td>
</tr>
<tr>
<td>Correct explanation &amp; computational estimation</td>
<td>67 (21.3%)</td>
<td>48 (15.3%)</td>
<td>10 (3.5%)</td>
<td>19 (6.7%)</td>
</tr>
<tr>
<td>(59.3%)**</td>
<td>(33.1%)**</td>
<td>(58.8%)**</td>
<td>(30.2%)**</td>
<td></td>
</tr>
<tr>
<td>Wrong explanation</td>
<td>43 (13.7%)</td>
<td>43 (13.7%)</td>
<td>79 (28%)</td>
<td>30 (10.6%)</td>
</tr>
<tr>
<td>No answer</td>
<td>314 (100%)</td>
<td>314 (100%)</td>
<td>282 (100%)</td>
<td>282 (100%)</td>
</tr>
</tbody>
</table>

Table 4 presents the written explanation of the students’ way of thinking, indicating that 30% to 50% of the total sample did not explain their mental processes at all.

The majority of the students (68.6% to 91%) who used computational estimation were able to produce adequate and correct justification of their way of thinking in each of the four problems. The smallest percentage (68.6%) of them that provided an explanation appears to problem 2.P.6., which required an addition of the amounts; the acts of the rounded amounts were performed by many of them nevertheless, without giving any explanations.

We observe that the behavior of students who perform exact computation is different in the written explanation of the solution. Almost half of these students justify their solution in the problem with the sum of the fractions (1.P.5., 59.3%) and the problem with percentage calculation (1.P.6., 58.8%). Less than half of the students justify their solution in the problem with multiplication of two numbers (2.P.5., 33.1%) and the problem with addition of decimals (2.P.6., 30.2%). Finally, most of the students who calculate the solution accurately perform arithmetic operations without however justifying and explaining the way they did it.

2.3.5. Comparison between computational estimation and metacognition ability with problem solving ability

Comparison between computational estimation ability and problem-solving ability
To compare students’ computational estimation ability to their problem-solving ability, we correlated success in computational estimation with problem-solving

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4 * The percentage which corresponds to students who executed correct computational estimations is marked in bold.
5 ** The percentage which corresponds to students who produced exact calculation correctly is marked in bold.
performance. As problem solving performance we consider the students’ performance in solving the three remaining problems of the examination.

For each problem success were marked with 2.5 points and the maximum score achieved by the three problems was 7.5 units. In the following table (Table 5), we present and compare by t-test the means and standard deviations of performance in the three problems of the test in terms of those students who used computational estimations and those who did not.

Table 5: Computational estimation and problem solving ability comparison

<table>
<thead>
<tr>
<th></th>
<th>1.P.5. 1/2plus 3/8</th>
<th>2.P.5. 28 x 56</th>
<th>1.P.6. 9.84% of 816</th>
<th>2.P.6. 1.26+4.79+0.99 +1.37+2.58</th>
</tr>
</thead>
<tbody>
<tr>
<td>Students who use computational estimation</td>
<td>M= 4.61, SD=2.12</td>
<td>M=5.44, SD=2.17</td>
<td>M=5.52, SD=1.93</td>
<td>M=5.06, SD=2.08</td>
</tr>
<tr>
<td>Students who don’t use computational estimation</td>
<td>M=3.44, SD=1.99</td>
<td>M=3.3, SD=1.81</td>
<td>M=3.49, SD=2.34</td>
<td>M=3.37, SD=2.41</td>
</tr>
<tr>
<td>Comparison</td>
<td>t=4.603, df=312, p&lt;0.001</td>
<td>t=8.329, df=312, p&lt;0.001</td>
<td>t=7.331, df=280, p&lt;0.01</td>
<td>t=6.285, df=280, p&lt;0.01</td>
</tr>
</tbody>
</table>

The 5th grade students’ mean in problem solving is 3.77 whereas in the 6th grade it is 4.19.

For example, in the 1.P.5. problem the means and standard deviations of performance in the three problems of the test in terms of those students who used computational estimations are M=4.66, SD=2.12 and those students who did not use computational estimations are M=3.44, SD=1.99. The difference of the means of performance in the three problems of the test in terms of those students who used computational estimations and those who did not is statistically significant (t = 4.603, df = 312, p < .001).

In table 5, the comparison of the means indicated that there is a statistically significant difference in problem solving performance between the students who used computational estimation and those who didn’t. This fact signifies that students who used computational estimation were better “problem-solvers”.

Comparison of metacognitive ability and problem solving ability
At this point, students’ metacognitive and problem-solving ability is correlated. Namely, we compared the means of their performance in the three problems in which the students provided a written explanation of the way they computed (of those having metacognitive ability) and those who did not explain their computations.
Table 6: Comparison of metacognitive ability in computational estimation and problem-solving ability

<table>
<thead>
<tr>
<th></th>
<th>1.P.5. 1/2plus 3/8</th>
<th>2.P.5. 28 x 56</th>
<th>1.P.6. 9.84% του 816</th>
<th>2.P.6. 1.26+4.79+0.99+1.37+2.58</th>
</tr>
</thead>
<tbody>
<tr>
<td>Students who explained their solution</td>
<td>M=4.7 SD=1.96</td>
<td>M=4.81 SD=2.16</td>
<td>M=5.63 SD=1.93</td>
<td>M=5.39 SD=2.05</td>
</tr>
<tr>
<td>Students who did not explain their solution</td>
<td>M=2.89 SD=1.67</td>
<td>M=3.26 SD=1.91</td>
<td>M=3.87 SD=2.42</td>
<td>M=3.7 SD=2.17</td>
</tr>
<tr>
<td>Comparison</td>
<td>t=-7.615 df=249 p&lt;.01</td>
<td>t=-6.152 df=265 p&lt;.01</td>
<td>t=-5.296 df=171 p&lt;.01</td>
<td>t=-6.016 df=225 p&lt;.01</td>
</tr>
</tbody>
</table>

According to the means comparison with t-test presented in table 6, there was a statistically significant difference between the problem-solving performance of students who explained their solutions in computational estimation and those who didn’t. For example, in the 1.P.5. problem, the means and standard deviations of their performance in the three problems for which the students provided a written explanation of the way they computed are M=4.7 and SD=1.96, and for those who did not explain their computations are M=2.89 and SD=1.67. The difference of the means of performance in the three problems in which the students provided a written explanation of the way they computed and those who did not explain their computations is statistically significant (t = 7.615, df = 249, p < .01).

Students of both classes who displayed metacognitive ability by explaining their solution strategies in problems of computational estimation were better “problem-solvers”.

3. CONCLUSION-DISCUSSION

As we have already mentioned, the sample consisted of students with a positive attitude towards mathematics, because they participated voluntarily in the competition. In addition, as we can conclude for the results of this study, they were not systematically trained in computational estimation at their school and did not have the opportunity to use specific strategies for these calculations; therefore, their strategies were based on personal conceptions, and in that sense they were spontaneous and self-developed.

The results of this study showed that most students calculated computational estimation problems with exact computation. These results are in accordance with Sowder’s (1992) claim ‘poor estimators seem to be bound, with only slight variations, to one strategy—that of applying algorithms more suitable for finding the exact answer’ (p. 375).
In a research by Lemonidis & Kaimakami (2013) conducted with 50 Greek pre-service elementary teachers in computational estimation the same problems were posed, and the percentages of those using computational estimation were higher than those of the students of this research: 1.P.5. 88% (vs. 28.3%), 2.P.5. 70% (vs. 22.3%), 1.P.6. 60% (vs. 34.4%), and 2.P.6. 80% (vs. 48.6%).

Students of this sample, were not aware of the special computational estimation strategies like the special numbers strategy, the rounding with compensation and the front-end strategy. Their errors also illustrated their attachment to the written algorithms, which they falsely tried to execute mentally. It is also clear that a great number of students made errors in their attempt to perform an estimation, which reveals their low capacity about computational estimation. This is evidence of informal knowledge which does not derive from the school. This lack of estimation strategies supports the findings of Reys et al. (1991b), who state that ‘estimation skills do not necessarily evolve from the development of traditional written computation algorithms’ (p. 55). This is also consistent with the belief that computational estimation skills are highly dependent on number sense, and that the absence of estimation skills is due to a lack of number sense (Reys et al., 1991b; Sowder, 1992; Markovits & Sowder, 1994; Yang, 2005). Students don’t know well the number sense strategies for estimation and this is a result shown by mant researches (Reys et al., 1991b; Sowder, 1992; Markovits & Sowder, 1994; Yang, 2005).

In the same problems, Greek pre-service elementary teachers (Lemonidis & Kaimakami, 2013) used the above strategies on higher percentages. More precisely the special numbers strategy is used by 84% (vs. 28%) in 1.P.5. problem and 54% (vs. 20.9%) in 1.P.6. Rounding and compensation strategy in 1.P.5. problem is used by the 64% (vs 5.4%) and finally the front-end strategy in 2.P.6. problem is used by the 46% (vs. 7.8%) of teachers.

As far as their written expression is concerned, we saw that many students did not write down the way they were thinking when they were trying to solve a problem. These students had not been taught and trained in this metacognitive behavior. On the other hand, students who used computational estimation wrote down and justified their reflections more than those who computed exactly. This result can be explained maybe by the fact that students who use computational estimation have a better understanding of the problem and of calculations used and they can write down and justify their reflections. The above result is consistent with the conclusion reached by Sowder (1992) after the review of many researches: ‘Yet there is one clear message that comes through several of these studies. Good estimators are flexible in their thinking, and they use a variety of strategies. They demonstrate a deep understanding of numbers and operations, and they continually draw upon that understanding.’ (p. 375)

In the case of problem-solving and computational estimation ability relation, it was found that students who used computational estimation were better “problem-solvers” than those who did not use computational estimation. This result agrees with

This was also the case of students who wrote down their way of thinking in comparison to those who did not. Students, who had the ability to switch methods, informally used the computational estimation and were able to write down their thoughts, had metacognitive ability and were therefore better problem-solvers.

To conclude, we can relate problem solving and metacognitive ability to the students’ flexibility in switching methods, and their inventiveness to use methods, which they had not actually been taught at school.

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**BRIEF BIOGRAPHIES**

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FIFTH AND SIXTH GRADE STUDENTS' NUMBER SENSE IN RATIONAL NUMBERS AND ITS RELATION WITH PROBLEM SOLVING ABILITY

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ABSTRACT
Problem solving and number sense are two of the core subjects on which strong emphasis is given in contemporary mathematics curricula of compulsory education. In this study, we examined fifth and sixth grade Greek students' number sense concerning the mental calculation with rational numbers and specifically fractions and percents. We attempt to analyze the behaviors of fifth and sixth graders in mental calculations with fractions and percent examining the performance of students, categorizing the mental strategies used by them. Despite of the educational importance of these two mathematical areas – problem solving and number sense in mental calculations with rational number – there are not studies which examine directly the relation of students’ performance in these two areas. This study has shown that the majority of students use rule-based strategies in operations with fractions and percents. Another result is that the students’ strategy choice (number sense or rule based) relates to their performance in problem solving.

Keywords: number sense, rational numbers, problem solving.

1. INTRODUCTION
The rational number concept is one of the most important, but also more complex mathematical concepts that children will experience during their primary school years. Empirical studies conducted in different countries have highlighted the difficulties faced by students. Students seem to respond well to the use of algorithms, but they lag behind in understanding the concept of fractional numbers and solving verbal and realistic problems involving fractions (Aksu, 1997; Dufour-Janvier, et al., 1987; Kerslake, 1986; Lesh, et al., 1987; Mack, 1995; Nunes, and Bryant., 2009;
Stafylidou & Vosniadou, 2004; Thompson, & Saldanha, 2003; Vamvakoussi & Vosniadou, S., 2010).

An integral part of number sense is the mental calculation in general and mental calculation with rational numbers in particular (Reys, R. 1984, Reys, B. 1985). Mental calculation with rational numbers contributes to a deeper understanding of rational numbers and their operations. Mental calculations based on conceptual understanding of numbers and operations and on a holistic approach to numbers (McIntosh, 2006). In this research, we examined a sample that consisted of Greek students in fifth and sixth grade primary school. Although Greek curricula include mental calculations with whole numbers, however, they only make a general reference and a piecemeal presentation of mental calculations with rational numbers, with emphasis on estimation procedures. Greek curricula and textbooks do not provide a specialized teaching proposal regarding mental calculations with rational numbers. In addition, teachers do not know the strategies as they are not trained in this particular subject. Therefore, it is important to investigate the Greek students’ number sense, regarding mental calculations with fractions and percents and study students’ strategies and their errors in operations with fractions and percents.

Yang (2005) in his research with sixth grade Taiwanese students, regarding the number sense for whole and decimal numbers found that these students were inclined to use ‘rule-based methods’ or ‘could not explain’ when responding to interview questions. They didn’t know how to explain their methods. In our research, in each task, we ask students to explain in writing how they have calculated. We do this in order to understand the strategies of students, but also to examine their ability to understand the operations which they have used. Problem solving has occupied a central position in school mathematics curricula and many studies have been conducted on this subject (Schoenfeld, 1992, 2007). Many researchers underline the correlation between mental calculation and problem solving (Reys, 1984, Thompson, 1999). However, there has not been specific research investigating the relationship between students’ ability in problem solving and their performance in mental calculations with fractions and percentages.

There is a research done by Louange, & Bana, (2010) with Year 7 students, which aim was to determine the relationship between students’ number sense and their problem-solving ability by means of paper-and pencil tests, classroom observations, and interviews of students and teachers. The results revealed a strong correlation between these two aspects of school mathematics. Wai and Kheong (1998) examined the correlation between problem-solving ability, mental calculation ability and estimation ability of primary 5th grade pupils in Singapore. Results from quantitative analyses showed that the scores of problem solving, mental calculation and estimation were significantly correlated.

In this paper, we try to contrast the ability in mental calculations with fractions and presents, with the problem-solving ability.
2. THE STUDY

The purpose of the present study is twofold: a) to examine the students’ number sense concerning mental calculations with fractions and percents, namely, to examine the performance and errors of students in operations, to record the strategies which the students use when performing mental calculations with fractions and percents, and b) to examine the relation between students’ mental calculations ability with fractions and percents and their problem solving ability.

2.1. Number sense in fractions and percents

What is number sense? It has been more than half a century since the concept of number sense was delimited (Dantzing, 1954) referring to the abilities of use and understanding of numbers, arithmetic relations and operations. Nevertheless, the number sense is difficult to define, so various researchers have identified several combinations that include a conceptual understanding of the number and specific numerical skills as components of number sense (Berch, 2005). Quoting McIntosh, Reys, & Reys, (1992), “Number sense refers to a person’s general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful strategies for handling numbers and operations.” (p. 3).

However, several researchers have reasoned that many students in the middle grades experience problems with the different aspects of number sense (Reys & Yang, 1998; Van den Heuvel-Paanhuizen, 1996, 2001; Yang, 2005; Verschaffel, Greer & DeCorte, 2007).

McIntosh & Dole (2000) have found that students can calculate accurately, mentally, without understanding, that is, high performance in mental calculation can be achieved without accompanying number sense. It is likely that these students, after much practice, can effectively use a strategy mechanically. Oi McIntosh et al. (1992) claim that high ability in written calculations is not necessarily accompanied by number sense. That is that, a student or an adult can find mechanically the correct answer for an operation without understanding the meaning of the numbers or the operation.

Caney and Watson (2003) conducted a study with 24 students, from grades 3-10, in order to record the strategies they used in performing mental calculations with fractions, decimals and percents. Despite the fact that the sample of students was relatively small, many mental strategies were exhibited in their effort to answer the problems. Many strategies used in solving part-whole number problems seem to coincide with those recorded in whole numbers. Some strategies employed with whole numbers, were not observed in operations with rational numbers. New strategies that were implemented when working with part-whole type numbers have also been recorded.

Callingham and Watson (2004) conducted a study (with a sample of 5,535 students) in order to describe a developmental scale of students' performance in
mental calculations with fractions, decimals and percents. The six levels of developmental scale seem to indicate an increasing understanding of the structure of part-whole numbers and the application of number knowledge gained in whole number contexts (factors, multiples, and place value).

2.2. Number sense or rule-based strategies for rational numbers

Strategies used by students in solving problems involving operations with rational numbers have been documented by several studies. These strategies are divided into conceptual and procedural (Clarke, & Roche, 2009; Post, Cramer, Behr, Lesh, & Harel, 1993; Yang, Reys, & Reys, 2009). Yang et al. (2009) refer to rule-based strategies and number sense-based strategies. In this study we adopt the terms number sense-based and rule-based strategies. Specifically:

Number sense strategies: These strategies are not taught in school, but they arise from the students’ ability to deal with rational numbers holistically based on conceptual characteristics of number sense. In number sense strategies include:

a. Residual thinking (Post and Cramer, 1987). When comparing $\frac{3}{4}$ and $\frac{7}{9}$ a student may think that $\frac{3}{4}$ has a residual of $\frac{1}{4}$ or $\frac{3}{8}$. Consequently the residual for $\frac{7}{9}$ ($\frac{2}{9}$) is less than the residual for $\frac{3}{4}$ ($\frac{2}{8}$). The fraction with the smaller residual is the bigger fraction.

b. Benchmarking (or transitive) (Post et al. 1986). When benchmarking, a student may compare a fraction to another well known fraction, such as $\frac{1}{2}$, or to a whole number (0 or 1). For example, when comparing $\frac{5}{8}$ and $\frac{3}{7}$, $\frac{5}{8}$ is bigger than $\frac{1}{2}$, and $\frac{3}{7}$ is smaller than $\frac{1}{2}$, therefore $\frac{5}{8}$ is bigger.

c. mental picture. For example, in substraction $\frac{3}{4} - \frac{1}{2}$, a student divides an imagined picture of rectangle into 4 parts (Caney and Watson, 2003) and

d. converting a fraction or percentage to a decimal.

Rule-based strategies: When using rule-based strategies students are based on memorized rules which are not necessarily linked to deep conceptual understanding. Rule-based strategies include: finding equivalent fractions with a common denominator (for adding, subtracting or comparing fractions), cross- multiplying fractions, applying memorized rules as: “In order to divide two fractions, copy the first fraction, invert and multiply the second fraction.”

3. METHOD

3.1. Sample

The sample consisted of 462 fifth and sixth grade students who participated in the sixth competition of "Nature and Life Mathematics", conducted in six towns of Western Macedonia, Greece. 290 students were fifth graders and 172 were sixth graders. Students who took part in this competition were not selected as their participation in the competition was completely voluntary. Their positive attitude
towards mathematics was perhaps their most distinctive feature. Although the participation in the competition is voluntary, students’ performance in mathematics is expected to be higher than the mean performance.

Finally, the students were not taught mental calculation strategies with rational numbers. Thus, the number sense strategies they used were spontaneous and self-developed.

3.2. Procedure

The competition took place out of school hours. The examination lasted one hour and was conducted in writing while the worksheet consisted of two mental calculation tasks and three word problems. In each mental calculation task, the students were asked to provide two modes of solution and to describe how they thought in writing.

2.3. The tasks

Every student of fifth and sixth grade had to answer five tasks, from which both refer to mental calculations with rational numbers and the other three tasks were word problems.

The tasks on mental calculations with rational numbers are listed below:

5th grade
Q51: I calculate with my mind: 1 - ⅓. I use two ways to answer. Every time I write the way I thought.
Q52: I calculate with my mind: 1/2:1/4. I use two ways to answer. Every time I write the way I thought.

6th grade
Q61: I compare the fractions 3/7 and 5/8. Which is bigger? I use two ways to answer. Every time I write the way I thought.
Q62: I find the 90% of 40. I use two ways to answer. Every time I write the way I thought.

The three problems for 5th grade were the following:
1. Easter excursion: In an Easter excursion involved 96 people, men, women and children. Men and women were totally 64. Women and children were totally 65. How many men took part in in the excursion, how many women and how many children?
2. The sculptor: Mr. Nick, the sculptor, bought four marble slabs. Each of them had a length of 2.5 meters. How many slabs of one meter can be cut from them?
3. The square: Peter argues that a shape is square when it has four sides which are pair wise parallel. Do you agree or disagree with Peter? Why? You can use shapes in your answer.

The three problems for 6th grade students were the following:
1. **Home distance**: Elena and Niki go to the same school. Elena lives within a distance of 17 km from the school and Niki’s home is located 8 kilometers away from the school. How many kilometers do they live away from each other?

2. **Kittens**: A cat has 6 kittens: one black, one white, one beige, one white - black, one white-beige and one black-beige. Maria chose three so that randomly two of them have at least one common color. How many different choices can we have?

3. **Pyramid**:
The number of cubes placed on each level of the pyramid depends on a rule.
The table shows the number of cubes at the first 3 levels.

<table>
<thead>
<tr>
<th>LEVEL</th>
<th>CUBES</th>
</tr>
</thead>
<tbody>
<tr>
<td>1ο</td>
<td>1</td>
</tr>
<tr>
<td>2ο</td>
<td>4</td>
</tr>
<tr>
<td>3ο</td>
<td>9</td>
</tr>
<tr>
<td>4ο</td>
<td>?</td>
</tr>
</tbody>
</table>

a) Find out how many cubes are there at the 4th level of the pyramid. Write all your thinking process.

b) How many cubes will there be at the 10th level of the previous pyramid? How do you know it? Write all your thinking. Write the rule by which you can find cubes for any level.

c) Is the rule that you found always true? Are you sure that this rule applies no matter how big you make the pyramid?

I am sure [ ] I’m not sure [ ]

3. **RESULTS**

3.1. **Performance**

In each task, the students were asked to provide the solution in two ways. The table below presents data on the students’ performance:

<table>
<thead>
<tr>
<th></th>
<th>Q51:1-1/4</th>
<th>Q52:½:1/4</th>
<th>Q61:3/7&amp;5/8</th>
<th>Q62:90%of 40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two correct answers</td>
<td>65 (22.5%)</td>
<td>52 (18%)</td>
<td>67 (39%)</td>
<td>82 (47.5%)</td>
</tr>
<tr>
<td>Only one correct answer</td>
<td>113 (39%)</td>
<td>80 (27.5%)</td>
<td>57 (33%)</td>
<td>56 (32.5%)</td>
</tr>
<tr>
<td>A correct and a wrong answer</td>
<td>25 (8.5%)</td>
<td>48 (16.5%)</td>
<td>15 (9%)</td>
<td>4 (2.5%)</td>
</tr>
<tr>
<td>Two wrong answers</td>
<td>28 (9.5%)</td>
<td>36 (12.5%)</td>
<td>10 (6%)</td>
<td>4 (2.5%)</td>
</tr>
<tr>
<td>A wrong answer</td>
<td>46 (16%)</td>
<td>66 (22.5%)</td>
<td>19 (11%)</td>
<td>17 (10%)</td>
</tr>
<tr>
<td>No answer</td>
<td>13 (4.5%)</td>
<td>8 (3%)</td>
<td>4 (2.5%)</td>
<td>9 (5%)</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>290 (100%)</strong></td>
<td><strong>290 (100%)</strong></td>
<td><strong>172 (100%)</strong></td>
<td><strong>172 (100%)</strong></td>
</tr>
</tbody>
</table>

MENON: Journal Of Educational Research [ISSN: 1792-8494]
http://www.edu.uowm.gr/site/menon

1" Thematic Issue
12/2014
As shown in Table 1, few students were able to calculate the operations in two ways. In the 5th grade their success was 22.5% and 18%, while in the 6th grade it was 39% and 47.5% respectively. Many students gave only one correct answer (39% and 27.5% of the 5th graders and 33% and 32.5% of the 6th graders). Several students failed to calculate the operations and gave one or two wrong answers or no answer at all (30% and 38% of the 5th graders – 19.5% and 17.5% of the 6th graders).

3.2. Strategies

Table 2 shows the percentages of students who use each strategy:

Table 2: Percentages of strategies used by students that answer one or two questions correctly

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Rule-based strategies</td>
<td>180 (67%)</td>
<td>185 (80.5%)</td>
<td>122 (58%)</td>
<td>140 (62.5%)</td>
</tr>
<tr>
<td>Number sense strategies</td>
<td>89 (33%)</td>
<td>45 (19.5%)</td>
<td>88 (42%)</td>
<td>84 (37.5%)</td>
</tr>
<tr>
<td>The number sense strategies</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Converting a Fraction or a Percent to a Decimal</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mental picture</td>
<td>7 (2.5%)</td>
<td>18 (8.5%)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Benchmarks to ½ or 10%</td>
<td>7 (3%)</td>
<td>7 (3.5%)</td>
<td>19 (8.5%)</td>
<td></td>
</tr>
<tr>
<td>Residual thinking</td>
<td>7 (3.5%)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reduction in unit</td>
<td></td>
<td></td>
<td>23 (10.3%)</td>
<td></td>
</tr>
</tbody>
</table>

Below are a few examples of these strategies:


- **Mental picture.** For Q51:1-1/4: “I see 1 as an entire pizza or a clock with four quarters. I remove the 1/4 and ¾ is left”. For Q52: 1/4 fits twice into ¾.

- **Benchmarks to one half.** For Q61:3/7&5/8: 5/8 is bigger than 1/2. 3/7 is smaller than 1/2. So 5/8 >3/7.

- **Residual thinking.** For Q61:3/7&5/8: 3/7 is 4/7 away from being a whole (7/7). 5/8 is 3/8 away from being a whole (8/8). Because 4/7>3/8, we have 5/8 > 3/7.

- **Reduction in unit.** For Q62:90% of 40: 1% of 40 is 40:100=0,4. So: 90x0,4=36.

As can be seen in Table 2, the majority of the students fled in the implementation of the rule – based strategies to find the result. Specifically, the percentage of the students who chose the rule – based strategies as a first or second option goes up to 67% in the subtraction 1-1/4 in 5th grade, 80.5% in the division 1/2+1/4 in 5th grade,
58% in comparing fractions (3/7 & 5/8) in 6th grade and 62.5% in finding the percentage 90% of 40 in 6th grade.

This behavior, namely the widespread use of rule – based strategies by the students, may be justified by the “didactic contract” (Brousseau, 1984), which is formed in classrooms where students are encouraged to use only one method (rule of operation).

The percentage of students who chose to convert fraction or decimal to percentage ranges from 16.5% to 30.5%. Conceptual strategies, as using mental representations, using the 1/2 as a reference point, and residual thinking, seem to be significantly less used.

**Number sense strategies**

As mentioned previously, the students were asked to give two solutions to each task. Table 3 below lists the types of strategy (number sense or rule-based) which the students, who were able to give two correct answers, chose and the selections of students who gave only one correct answer. As shown in Table 3, the vast majority of students who gave two ways of solution, chose one rule-based and one number sense strategy. Regarding the subtraction of fractions, this percentage reached 98.5%. It is interesting to note that the vast majority of pupils who were able to give only one correct solution used a rule-based strategy.

**Table 3: Strategies use by students who gave only one or two correct answers**

<table>
<thead>
<tr>
<th>Use of strategies</th>
<th>Q51: 1-1/4</th>
<th>Q52: 3/7 &amp; 5/8</th>
<th>Q61: 3/7 &amp; 5/8</th>
<th>Q62: 90% of 40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategies are used by Students who gave two correct answers</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 rule-based and 1 number sense</td>
<td>64 (98.5)</td>
<td>33 (62.5%)</td>
<td>61 (91%)</td>
<td>57 (69.5%)</td>
</tr>
<tr>
<td>2 rule-based</td>
<td>0 (0%)</td>
<td>18 (34%)</td>
<td>0 (0%)</td>
<td>19 (23%)</td>
</tr>
<tr>
<td>2 number sense</td>
<td>1 (1.5%)</td>
<td>2 (3.5%)</td>
<td>6 (9%)</td>
<td>6 (7.5%)</td>
</tr>
<tr>
<td>TOTAL</td>
<td>65 (100%)</td>
<td>53 (100%)</td>
<td>67 (100%)</td>
<td>82 (100%)</td>
</tr>
<tr>
<td>Strategies are used by students who gave only one correct answer</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rule-based</td>
<td>99 (87.5%)</td>
<td>74 (95%)</td>
<td>43 (75.5%)</td>
<td>40 (71.5%)</td>
</tr>
<tr>
<td>Number sense</td>
<td>14 (12.5%)</td>
<td>4 (5%)</td>
<td>14 (24.5%)</td>
<td>16 (28.5%)</td>
</tr>
<tr>
<td>TOTAL</td>
<td>113 (100%)</td>
<td>78 (100%)</td>
<td>57 (100%)</td>
<td>56 (100%)</td>
</tr>
</tbody>
</table>

**3.3. Errors**

The following table categorizes the students’ errors. As shown, most errors are due to the students’ inability to implement the rule of operation correctly. Also, many of the students made errors, trying to implement a rule which corresponds to another arithmetic operation. For example, in subtraction of fractions (1-1/4), many of the 5th grade students inversed the numerator and the denominator of the second fraction and multiplied (influence of the rule of dividing fractions). In division of fractions (1/2+1/4) some 5th grade students, found a common denominator, divided the
Selecting numerators and kept the same denominator (clear influence of the rule of adding and subtracting fractions). Finally, some students in the 6th grade, when comparing two fractions (3/7 & 5/8) focused on the size of the numerator or the denominator only e.g. “between fractions 3/7 and 5/8, 5/8 is larger, because its numerator is larger or 3/7 is larger because its denominator is smaller”.

Table 4: Classification of errors

<table>
<thead>
<tr>
<th>Type of error</th>
<th>Q51:1-1/4</th>
<th>Q52:½÷1/4</th>
<th>Q61: 3/7&amp;5/8</th>
<th>Q62: 90%of 40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Misapplication of the rule of operation</td>
<td>68 (23.5%)</td>
<td>91 (31.5%)</td>
<td>12 (7%)</td>
<td>16 (9.5%)</td>
</tr>
<tr>
<td>Implementation of the rule of another operation</td>
<td>29 (10%)</td>
<td>48 (16.5%)</td>
<td>13 (7.5%)</td>
<td>6 (3.5%)</td>
</tr>
<tr>
<td>Focusing on the numerator or on the denominator only</td>
<td></td>
<td></td>
<td>19 (11%)</td>
<td></td>
</tr>
<tr>
<td>Other errors</td>
<td>23 (8%)</td>
<td>45 (15.5%)</td>
<td>3 (1.5%)</td>
<td>9 (5%)</td>
</tr>
</tbody>
</table>

3.4. Correlation between strategy selection and students’ problem solving ability

In order to determine if strategy selection (number sense or rule-based) is associated with the students’ problem solving ability, two groups of students were formed. The first group included students who used only number sense strategies and students who chose a rule-based and a number sense strategy, since as stated earlier, the students were asked to solve each exercise in two ways. In the second group students who chose rule-based strategies only were included. Thereafter, we compared the performance of the two groups in solving three word problems which were included on the worksheet of the competition. Student’s scores in problem solving ranged from 0 to 3. We performed t-tests and found significant differences between the two groups of students. Table 5 below presents the results of this comparison.

Table 5: Correlation between strategy selection and students’ problem solving ability

<table>
<thead>
<tr>
<th>Question</th>
<th>Number sense</th>
<th>Rule-based</th>
<th>t</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q51:1-1/4</td>
<td>1.54</td>
<td>0.82</td>
<td>0.99</td>
<td>0.68</td>
</tr>
<tr>
<td>Q52:½÷1/4</td>
<td>1.61</td>
<td>0.72</td>
<td>1.12</td>
<td>0.80</td>
</tr>
<tr>
<td>Q61:3/7&amp;5/8</td>
<td>1.48</td>
<td>0.52</td>
<td>1.04</td>
<td>0.54</td>
</tr>
<tr>
<td>Q62:90%of 40</td>
<td>1.51</td>
<td>0.50</td>
<td>1.05</td>
<td>0.59</td>
</tr>
</tbody>
</table>

As shown, the performance of the students who chose number sense strategies is significantly better than students who used rule-based strategies only. Probably the students’ ability to interpret the operations conceptually and see the rational
numbers holistically is a useful skill in problem solving that enhances their performance in this area.

4. CONCLUSIONS

The results of this research show that Greek students in fifth and sixth grade have very weak number sense, concerning fractions and percents. They use mostly rule-based strategies that means they use memorized rules to operate with fractions and percents. In addition, classification of errors shows that the students’ errors are due to the fact that the students have not developed intuitive knowledge and understanding concerning rational numbers, but they make errors while trying to implement a written algorithm without understanding its meaning and function. The majority of students, whereas asked to give two ways of calculation, could only give a strategy which, in most cases, was a rule-based strategy. These results are strikingly similar to the approaches used by many middle grade students in Australia, Kuwait, Sweden, United States and Taiwan. For example, Yang (2005) assesses the number sense of 6th-graders in Taiwan with a series of questions, related to whole and decimal numbers. Results indicated that, regardless of performance level, very few number sense strategies (e.g. using benchmarks, estimation or numbers of magnitude) were used. The evidence also revealed that Taiwanese students tended to apply rule-based methods and standard written algorithms to explain their reasoning. The study of Markovits & Sowder (1994) found that about 42% of seventh graders tried to use written methods (finding the common denominator or changing the fractions to decimals) when compared 5/7 and 5/9. About 25% of them didn’t know how to solve it or gave an incorrect answer. The study of Reys & Yang (1998) and Yang & Reys (2002) also found that there were a high percentage of sixth and eighth graders could not meaningfully compare fractions. This is due to they are lacking in number sense. These results are not surprising since, as mentioned above, the mental calculations with rational numbers are not included in the curricula and textbooks which give emphasis on written algorithms. This fact is reflected in the opinions and attitudes of teachers.

Moreover, the data showed that the students, who demonstrate a kind of flexibility in choosing mental computation strategy and use number sense strategies, which reveal a deeper understanding of the rational numbers concept, have better performance in solving mathematical problems compared to students who use rule-based strategies only. Similarly, Louange, & Bana, (2010) in their research on year 7 students find a strong correlation between number sense and their problem-solving ability. Finally, Wai and Kheong (1998) find significant correlation between problem-solving ability and mental calculation ability of primary 5th grade pupils in Singapore.

Wai and Kheong (1998) examined the correlation between problem-solving ability, mental calculation ability and estimation ability of primary 5th grade pupils in Singapore. Results from quantitative analyses showed that the scores of problem solving, mental calculation and estimation were significantly correlated.
Limitations of the study
There were several limitations in this research, among them the small number of operations presented to the pupils of both 5th and 6th grade. Another limitation was the lack of a realistic context for the operations suggested. We could assume that if the operations had been included within a realistic context, the students would have employed different strategies.

Educational implications
The results of the research suggest that the teaching of rational numbers in a way that would be more comprehensible would be beneficial for the Greek students. The use of mental calculations and estimation in the operations with rational numbers and the use of various strategies can help to the improvement of the teaching of rational numbers. Understanding the significance of the operations is most likely to reduce students' errors, which are caused by incorrect and mechanical, -without understanding- implementation of algorithms. It would be interesting to investigate the effects of an experimental teaching intervention with a focus on rational numbers, emphasizing the understanding, the use of mental calculation and the flexibility in operations on students' behavior.

REFERENCES


**BRIEF BIOGRAPHIES**

**Charalambos Lemonidis** is a professor of didactic of mathematics at the Department of Primary Education - University of Western Macedonia (Florina, Greece). He is born in Amygdaleona of Kavala in 1961. He has graduated from the Department of Mathematics - Aristotle University of Thessaloniki. He received a master and a Ph.D. in Didactic of Mathematics from the University of Luis Pasteur (France). He is director of the Postgraduate Program in the Department of Primary Education at the University of Western Macedonia. He has founded the school entitled “Mathematics of Nature and Life”. His scientific interests are, mental calculation, use of technology in teaching/learning mathematics, mathematics disabilities, life long learning of mathematics.

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COMPUTATIONAL ESTIMATION IN THE GREEK PRIMARY SCHOOL: TASKS PROPOSED FOR ITS TEACHING

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ABSTRACT

The goal of this study was to examine prospective primary school teachers’ views of computational estimation and its teaching in primary school. Data was gathered through questionnaires administrated to 113 Greek pre-service teachers regarding the value they accredit to computational estimation and their suggestions for teaching it followed by interviews of 10 pre-service teachers regarding their understanding about computational estimation and skills needed to be developed. Results revealed that the majority of prospective teachers believe in the importance of computational estimation for both daily life and school. In the tasks they propose for teaching computational estimation, they incorporate context that relies on children’s out-of-school experiences, indicating their willingness to relate these experiences to computational estimation and, in general, to mathematics. Educational implications are discussed with regards to the need for computation estimation to be further acknowledged and implemented in the classroom as well as instruction to be provided to teachers.

Keywords: computational estimation, primary school mathematics, prospective teachers, teacher education.

1. INTRODUCTION

Mental computation and computational estimation are two integral aspects of number sense (McIntosh 2004, Reys & Yang 1998) and are seen as important for making sense of numbers, operations and their relationships. Whereas mental computation produces exact arithmetic answers, computational estimation refers to arithmetic operations and to decisions related to mathematics computations that can be made about their results producing approximate outcomes. They both make use of mental procedures without external helps—such as paper and pencil. In computational estimation, however, data are transformed or substituted with simple
numbers, that is, numbers that are easy to remember and work with in mental arithmetic operations.

Mental computation has recently been included in the compulsory education curriculum for mathematics in Greece, as well as in many other countries. There are many reasons for teaching mental computation at school, such as its contribution to written arithmetic and problem solving (Vershaffel, Greer & De Corte 2007). One strong reason, though, for incorporating mental computation into the primary school curriculum is its relationship with computational estimation (Reys 1984). A good basis for computational estimation consists of having adequate skill with mental algorithms that are easy to memorize and quick to use. Thus, computational estimation can now be found in primary school mathematics and has mainly been considered as an important topic of mathematics education mainly for its practical utility (Segovia & Castro 2009) and for broadening the restrictive view of mathematics according to which mathematics was synonymous with written computations (McIntosh 2004).

Concerning effects on primary school children’s understanding of number and number sense development have been identified in the literature with regards to the dominance of the teaching and learning of conventional written algorithms in the classroom (Vershaffel et al 2007). Teachers allocate a lot of time teaching children written computations (McIntosh 2004) as well as strategies for written computations that require children to follow fixed sequential steps and certain procedures (Ghazali, Alias, Ariffin & Ayub 2010). This heavy emphasis on rules associated with written computation makes students rule followers and leaves little place for computational estimation to develop. This is obvious from research findings regarding children’s reliance on the use of algorithms. Alsawaie (2012), for example, found that high-achieving 6th grade children in the United Arab Emirates were highly dependent on school-taught rules that were confusing for them when solving arithmetic problems and discouraged them from developing number-sense based strategies. This finding is consistent with the findings of Yang (2005) with 6th grade children who were also affected deeply by the need to provide accurate operations, even when computational estimation was needed. This overemphasis on written computation using standard algorithms brings out a more pessimistic interpretation. It was the high-achievers in Alsawaie’s study (2012) who exhibited lack of number sense; therefore, one can imagine what this may mean for low mathematics achievers and how rule based methods hinder their development of mathematical thinking and reasoning. However, little persistence in adults’ use of formal written computation was found by Northcote and McIntosh (1999). In their survey, conducted with two hundred adults over a twenty-four-hour period, calculations that required only an estimate were used. It was only 11.1% of the participants who used written algorithms, whereas almost 85% of them performed computational estimations. This finding led Northcote and McIntosh to believe that it is the mathematics classroom that implies the use of conventional algorithms and discourages the application of estimation procedures.
Adults’ difficulty in estimation is reported by Hanson and Hogan (2000) who found that college students felt uncomfortable with the process of estimation. Their insistence on finding exact answers and their erroneous responses to estimation problems were explained by their lack of awareness of the meaning and the process involved in estimation, probably due to their limited exposure to estimation in schools. Lemaire, Arnaud and Lecacheur (2004) documented changes in computational estimation performance that are related to adults’ age differences, with older adults providing less accurate estimates than younger ones. Studies examined teachers’ performance on computational estimation problems (Desli & Anestakis 2014, Tsao 2013) and the strategies they employ (Lemonidis & Kaimakami 2013, Yang, Reys & Reys 2009, Segovia & Castro 2009) have shown their difficulty in estimation.

Research with teachers has lately been extended to involve examining their views of computational estimation. Alajmi (2009) investigated Kuwaiti mathematics teachers’ views about computational estimation and its significance in school. Although almost 70% of the teachers agreed that computational estimation is an important skill for everyday life, only 20% saw benefits from teaching it in the classroom. Most of those who believed in teaching computational estimation argued that it should be taught after students master standard algorithms giving priority on procedural algorithms over estimation. The important mathematics goal for these teachers was developing paper-and-pencil procedures. They mainly feared the effect of computational estimation on students’ development of algorithms for computing exact answers. It is interesting to mention that, at the time Alajmi’s study was carried out, computational estimation had not been included in the national curriculum in Kuwait; teachers’ missing training and lack of instruction on computational estimation may partly explain their fears. More promising results were revealed by Tsao (2013) in a study conducted with pre-service elementary teachers. In this study, participants generally held computational estimation in high regard and viewed it necessary, useful and beneficial. They stated little confidence in their computational estimation skills and did not particularly enjoy computational estimation. For the purpose of his study, Tsao (2013) also gathered pre-service teachers’ performance data on computational estimation, and found positive correlations with their views.

The fact that teachers’ content knowledge in mathematics and their beliefs concerning mathematics are strongly related to the way they teach mathematics has been acknowledged (Ball, Thames & Phelps 2008, Pajares 1992). With regards to computational estimation, teachers can play a crucial role in teaching it and lead children to value the importance of computational estimation and use it. For example, Tsao and Lin (2011) referred to a teacher with number sense and related knowledge of number sense who stimulated and guided the students to increase number sense. Similarly, Alajmi and Reys (2007) suggested that teachers who value the importance of reasonable answers also place great emphasis on reasonableness in their teaching by encouraging students to recognize reasonable answers. It is plausible to assume
that if teachers have little knowledge of computational estimation and its use, they will have difficulty helping children recognize its value and develop computational estimation.

The aim of the present study was to examine prospective primary school teachers’ views of computational estimation and its teaching in primary school. This study focused on computational estimation in primary school—in the scope of young children being educated—and was conducted with pre-service primary school teachers, trying to recognize their intentions before entering the classroom.

2. METHOD

Participants
This study was conducted, during the 2012-2013 spring semester, in a big-sized state university in Northern Greece with the pre-service teachers enrolled in a four-year elementary teacher education programme. After the completion of their studies, they were to be recruited as primary school teachers. Participants were 113 undergraduate students, predominantly female (86.7%), aged from 19 to 26 (mean age 21 years and 8 months). They were either attending their third year and were studying the course ‘Teaching Mathematics’ (64%) or they were in their fourth year and had already completed it (36%). Ten of them were randomly selected and further interviewed for the purpose of the study. No specific prior experience in computational estimation was required or examined other than that developed in the above mentioned course.

Data collection - Instruments of the Study
Two data collection methods were used, questionnaires and interviews, in order to examine prospective teachers’ views of computational estimation and its teaching in primary school.

The questionnaire used was part of a questionnaire used in a previous study on prospective teachers’ understanding of computational estimation and their strategies when dealing with computational estimation problems (Desli & Anestakis, 2014). The present study focuses on four questions that deal with: the importance of computational estimation in everyday life (question 1) and its teaching at school (question 2), skills that children must possess in order to develop computational estimation (question 3) and suggestions for posing a computational estimation task (question 4). To investigate prospective teachers’ thoughts about how to address computational estimation at school, the last question asked participants to suggest a task or activity that promotes computational estimation for primary school children. The first two questions were initially close-ended that needed further quantification, whereas the last two questions were open-ended. The participants scored in the first two questions on a five-point Likert-type scale (1=very important, ..., 5=not important at all) and were further able to comment on their choice. The open-ended questions—because of their nature—did not seem to influence the participants to respond one
way or another, and they have been shown to result in a broad range of responses that can be scored using scales developed by the authors. The questionnaires were distributed during participants’ classes and collected at the same time. Participation in the study was voluntary and anonymous and lasted about 10 minutes. Of those surveyed, 113 returned a completed questionnaire, yielding 86% participation response rate.

Extending the questionnaire, individual semi-structured interviews were conducted one-on-one with ten pre-service teachers randomly drawn from those pre-service teachers who had completed the questionnaire described above. Three central issues regarding computational estimation were included in the interview to reveal prospective teachers’ ideas for the meaning of computational estimation and its use in primary school mathematics. The three issues pertained to: a) awareness of the differentiation between mental calculation and computational estimation, b) teaching computational estimation in primary school, and c) skills needed to develop for successful computational estimation. The same questions were asked to all pre-service teachers who participated in the interview, with general probes used to encourage them to expand their answers. Rather than revealing to interviewees a desirable perspective on computational estimation and emphasis on its teaching, questions that framed ideas in a neutral light were intended to be asked. Interviews took place at the end of the school year by the first author and lasted approximately 15 minutes each. The authors independently coded transcripts of each interview and reached about 98% agreement.

**Data analysis**
The goal of selecting data from questionnaires and interviews was to understand how emerging themes prospective teachers discussed in interviews related to what they believed and answered in the questionnaires, as well as to further expand on their ideas and beliefs. In order to facilitate identification of traits, participants’ ideas from both questionnaires and interviews will be presented with regards to: (a) the meaning of computational estimation and its importance, (b) the skills needed for computational estimation, (c) the tasks proposed for teaching computational estimation, and d) computational estimation in school textbooks.

3. RESULTS

**a. The meaning of computational estimation and its importance.**
All prospective teachers were asked during the interviews to provide general descriptions about the meaning they attribute to computational estimation. It was found that all of them stated that the computation is done mentally. Six participants indicated that estimation calls not for an accurate answer but for an answer close to the accurate. Four participants seemed to confuse computational estimations with mental calculations and were not sure if the result in computational estimations is exact or approximate. Two prospective teachers focused on differences that may
appear in people’s computational estimations. As computational estimations reflect personal approaches and decisions related to mathematics computations, different estimations –more than one- may emerge. “Every child can use a different technique to reach a result” according to Zoi, a 4th year student. That explains a range of estimates that are considered to be correct solutions to a problem. Kyriaki, a 3rd year student agrees that “It is easier not to compute the exact result ... to escape from the correctness, to discard the belief that there is one and only one answer”.

Participants’ responses in the questionnaire showed that computational estimation was considered important in everyday life (M=1.41, SD=0.62) by the majority of them, with no significant differences between third- and fourth-year students (F(1,109)=.739, p=.392). Its usefulness was recognized mainly in settings involving purchases or transactions where there is no time for exactness and no calculator is available.

Similarly important computational estimation was considered with regards to its teaching in primary school (M=1.53, SD=0.76), with participants from different years of studies revealing similar responses (F(1,109)=1.642, p=.203). Apart from the practical utility of computational estimation found in everyday life, they were focused on the benefits of teaching computational estimation in the mathematics classroom: they specified that the significance of estimation lies in the development of a flexible way of thinking. According to their responses, estimations do not limit children’s thinking, because estimations “make children think beyond algorithms and rule-based methods” and guide them “how to think”. Furthermore, some participants claimed that estimations are pleasant for children and intrigue their interest whereas others associated estimations with strategies used for checking the correctness of a result. Table 1 displays the mean scores regarding the importance of computational estimation in everyday life and school for the 113 participants.

Table 1: Mean scores and standard deviations in pre-service teachers’ responses regarding the importance of computational estimation in everyday life and school

<table>
<thead>
<tr>
<th>Importance</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>How important are computational estimations in everyday life?</td>
<td>1.41*</td>
<td>0.62</td>
</tr>
<tr>
<td>How important is the instruction of computational estimation in primary school?</td>
<td>1.53*</td>
<td>0.76</td>
</tr>
</tbody>
</table>

*A five-point Likert-type scale was used ranging from 1=very important, ..., 5=not important at all

Interestingly, 10 respondents (9%), coming from both third and fourth year groups, argued that teaching computational estimation in school is not important. One of them did not consider it important for everyday life either: he envisioned exactness as the only way “to ensure justice in everyday transactions”. Although the rest 9 prospective teachers viewed computational estimation as an important skill for everyday life, they did not see it as beneficial for doing mathematics in school. Some of them overemphasized the importance of exactness and suggested that priority should be given to methods that lead to “correct” answers, implying that estimations
do not produce valid results and are confusing for students. With regards to the latter, the idea that computational estimation is a confusing topic can be seen in a participant’s comments: “it is not clear ... children do not need such complicated calculations”.

A thought-provoking statement was made by a 4th year student who did not consider computational estimation as an important skill for everyday life, “though facilitating”, because “calculators and mobile devices offer exact, quick and effortless results”. However, she considered its teaching to be very important, implying that estimation problems challenge and improve children’s thinking. According to her, “children are asked to think about mathematics; to have sense of number magnitudes, to handle numbers and operations with ease and fun and to have a sharp eye for understanding mathematics”.

The importance of computational estimation was also revealed in the interviews with respect to its usefulness. Zoi, a 4th year student, claimed: “Computational estimations are useful in daily life issues. I am a person who uses computational estimations every day: at the supermarket; to calculate my pocket money; [to estimate] how much I will spend in a day... I use it very often. Not only with units of measurement like euros, but also with quantities. In recipes, for example, to estimate the flour needed -as a cook! Once you can use numbers, you can estimate as well!”. She sees estimations as embedded in everyday cultural practices. Such a statement was provided by almost all the interviewees. Nine of them highlighted the usefulness of computational estimations in settings involving money (e.g. when buying something). Participants did not only stress the practical utility of estimation but also its relation to number sense and flexibility. Phrases related to the flexible character of estimation, such as “creativity”, “fantasy”, “freedom to think our own way” “ease in working with numbers and operations” were extensively used by participants in order to verify the importance they attribute to computational estimation.

Based on the practical utility of computational estimation, all prospective teachers agree that estimation is a useful tool to check for the reasonableness of answers. It can provide insight into the order of magnitude of the answer. Four of them imply that estimation should be done before the calculation, so that the exact result is expected close to the estimation and can be checked for its correctness. Additionally, two prospective teachers imply that estimation should be done after the calculation as a verification method. The other four did not clarify the stage at which the estimation should be placed, before or after the calculation.

b. Skills needed for computational estimation.
In the questionnaire, participants were asked to name the skills that children must possess in order to develop computational estimations. A bottom-up content analysis of participants’ responses resulted in the categories presented in Table 2. Number sense and its components were the most common references made. In particular, knowledge of numbers, operations and their properties were mainly mentioned as components of number sense, because computational estimation requires, according
to a 3rd year student, “understanding of operations”. Other aspects of number sense, such as “relative number size” and “the ability to compare numbers”, were also mentioned. Last but not least, references to benchmarks, like “the concept of \( \frac{1}{2} \)”, and to the skill to decompose and recompose numbers were also included in this category.

Rounding strategy was cited by many participants (29 out of 113) as a prerequisite for computational estimation. Critical thinking skills were considered important as well, as it can be seen in a prospective teacher’s (3rd year student) comment: “critical thinking is necessary so that students understand that approximate answers are appropriate, logical and can stand”. Flexibility was also referred as “fantasy and adaptability” and “creativity to make alternative strategies up”, among participants’ comments. However, the feature –among others- that is considered implicit in computational estimation and was highlighted by the majority of the interviewees was the quickness of the estimate. As Sophie, a 4th year student said, “When people are in a hurry, computational estimations are very useful to reach an [approximate] result and help them deal with many situations, like buying things at the supermarket”.

Table 2: Frequency of prospective teachers’ references to the skills needed for computational estimation

<table>
<thead>
<tr>
<th>Skills needed</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number sense and components</td>
<td>68</td>
</tr>
<tr>
<td>Rounding strategy</td>
<td>29</td>
</tr>
<tr>
<td>Critical thinking</td>
<td>20</td>
</tr>
<tr>
<td>Flexibility</td>
<td>14</td>
</tr>
<tr>
<td>Logic</td>
<td>10</td>
</tr>
<tr>
<td>Place value</td>
<td>8</td>
</tr>
<tr>
<td>Quickness</td>
<td>8</td>
</tr>
<tr>
<td>Mental calculations</td>
<td>7</td>
</tr>
<tr>
<td>Spatial &amp; quantitative reasoning</td>
<td>6</td>
</tr>
</tbody>
</table>

Note: Participants were allowed to make more than one reference

An alignment of participants’ responses when completing the questionnaire and being interviewed was demonstrated. The interviewees proposed the same skills as above. Number sense, in general, was implied in the pre-service teachers’ interviews. Zoi, a 4th-year student, explained: “Students need to understand how the four operations work, to handle them with ease, to master them... to know them very well, maybe automatically, to have their own tricks or techniques in order to compute quickly. In a multiplication, it is the same to have 2x20 as 4x10? Being able to do this stuff, to play with numbers... and to understand how relative all these number sizes are... 20 can break down into two tens. This is how estimation works.” Other skills, such as rounding, logic, critical thinking and mental computations were mentioned as well.
Interestingly, half of the participants regarded conceptual understanding of the four operations as the most significant skill needed for computational estimation. Dimitra, a 3rd-year student, insisted: “The child needs a general mathematical thinking. She must acquire the concept of addition, subtraction, multiplication and division. She must have all of these in mind, that is, the meaning of the operations”. Procedural knowledge is important, too, as Kyriaki, a 3rd-year student, explains: “When I solve [computational estimation] problems I often use different techniques from problem to problem, and these techniques help me solve all problems. Sometimes techniques are necessary. I discover them myself, but some had been taught to me... Thus, this skill [procedural knowledge] is important indeed, but I think it can be gradually discovered by the child... It emerges from the practice”.

c. Tasks proposed for developing computational estimation.
As part of the questionnaire, over half of the students (62 out of 113) suggested tasks that could be used in the primary school in order to promote primary school children’s computational estimation. These tasks were initially analyzed according to the type of estimation they referred to (see Table 3). More than half of the tasks (56.5%) referred to computational estimation, such the one proposed by a 4th-year student: “Can you estimate whether the sum 3,93+5,68+1,02 is smaller or bigger than 10?”. Other estimation types, like measurement estimation, were also found. A 3rd year student noted: “Which of the following buildings is taller? The number of floors and their heights between floors are given”. One numerosity estimation task was proposed by a 3rd year student (“Can you estimate about how many students attend classes at your school?”) and one number line estimation task by another 3rd year student (“Place the following fractions on the number line:...”). Participants’ references to types of estimation –different to computational- suggest their difficulty to distinguish between computational estimation and other types of estimation. This difficulty can be of great interest for future teacher training.

However, there were tasks that could not be classified as estimation tasks because either no sufficient information was provided (lack of task elements category), or estimation was not required for its solution (estimation not necessary category). The latter category included irrelevant tasks (for example, a 3rd year student suggested: “two cars can fit in this road, how many trucks can fit in?”) or tasks that require exact computation.

<table>
<thead>
<tr>
<th>Types of estimation</th>
<th>Frequency</th>
<th>Percent (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computational estimation</td>
<td>35</td>
<td>56.5</td>
</tr>
<tr>
<td>Numerosity estimation</td>
<td>1</td>
<td>1.6</td>
</tr>
<tr>
<td>Measurement estimation</td>
<td>4</td>
<td>6.5</td>
</tr>
<tr>
<td>Number line estimation</td>
<td>1</td>
<td>1.6</td>
</tr>
<tr>
<td>Lack of task elements</td>
<td>11</td>
<td>17.7</td>
</tr>
<tr>
<td>Estimation not necessary</td>
<td>10</td>
<td>16.1</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>62</strong></td>
<td><strong>100</strong></td>
</tr>
</tbody>
</table>
The tasks suggested by the participants were further analyzed according to the context that surrounded them, for the purpose of potentially revealing some of the most frequently present contexts. A top-down content analysis was undertaken in order to classify the context in the tasks, following Wager’s (2012) framework for incorporating out-of-school practices in mathematics, according to which students’ cultural and out-of-school experiences can be related to school mathematics.

The analysis resulted in the following categories:

- **non-context tasks**: this category involved mathematical problems with the absence of surrounding context (e.g. ‘which sum is closer to 100,000?’).
- **out-of-school context**: tasks in this category use students’ experiences as a context for word problems. No embedded practices were identified.
- **out-of-school activities related to mathematics**: in this category out-of-school activities are matched to school mathematical practice and students’ informal strategies or embedded practices are not taken into account.
- **embedded cultural practices**: in this category the context drives the mathematics in a way that a particular interest with an everyday life situation may be used in order to create an activity that has a particular mathematical content. Informal strategies students use in everyday activities are taken into consideration.
- **teacher initiated situation setting**: this category involves tasks that the school/teacher has developed and all students share.

Tasks that involved context (35 out of 41) were many more than non-context tasks. Most context tasks (22 out of 35) included out-of-school activities related to mathematics. Tasks with embedded cultural practices, such as estimating the cost of a birthday present for each child depending on the number of children going to the birthday party, also emerged. Similar were the instances that teacher initiated settings which lead to shared experiences for all children were suggested, such as the simulation of a street market in the classroom or a visit to the super market. The five categories related to context and their frequencies are shown in Table 4, whereas Table 5 presents with examples of the tasks separately for each category.

**Table 4**: Types of context found in the tasks proposed by participants

<table>
<thead>
<tr>
<th>Types of Context</th>
<th>Frequency</th>
<th>Percent (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-context tasks</td>
<td>6</td>
<td>14,6</td>
</tr>
<tr>
<td>Out-of-school context</td>
<td>9</td>
<td>22</td>
</tr>
<tr>
<td>Out-of-school activities related to mathematics</td>
<td>22</td>
<td>53,8</td>
</tr>
<tr>
<td>Embedded cultural practices</td>
<td>2</td>
<td>4,8</td>
</tr>
<tr>
<td>Teacher initiated situation setting</td>
<td>2</td>
<td>4,8</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>41</strong></td>
<td><strong>100</strong></td>
</tr>
</tbody>
</table>
Table 5: Examples of context tasks proposed by participants by type of context

<table>
<thead>
<tr>
<th>Types of Context</th>
<th>Story</th>
<th>Example</th>
</tr>
</thead>
</table>
| Out-of-school context (9) | Estimation of food mass (2)  
                          | Estimating the cost of products (2)  
                          | Ordering planet distances from sun (1)  
                          | Estimating the distance run (1)  
                          | Comparing different parts of the same chocolate bars (1)  
                          | Comparing height in a block of flats (1)  
                          | Estimating calories in a portion of food (1)  
                          | A 500g package of spaghetti has 1892kcal. How many kcal is a portion of spaghetti if the package serves for about 4 persons? |
| Out-of-school activities related to mathematics (22) | Student buying things using his pocket money (2)  
                          | Calculator button has broken (2)  
                          | Estimating the number of students in a school (1)  
                          | Comparing prices at discounts (1)  
                          | Choosing the route with the smallest travel cost (1)  
                          | Estimating if we can reach a destination by car on time (speed given) (1)  
                          | Estimating the amount to be paid in advance (1) or at a gas station (for oil and gas) (1)  
                          | Estimating monthly expenses (1)  
                          | Estimating time intervals (2)  
                          | Shopping at the supermarket (8)  
                          | Shopping at a kiosk (1)  
                          | Helen wants to compute 33X999, but the “9” button in her pocket calculator is broken. How can she estimate the result with the mind?" |
| Embedded cultural practices (2) | Estimating the cost of a birthday present for each child depending on the number of children who attend the birthday party (1)  
                          | Estimating the cost of a school festival (1)  
                          | Our class is organizing a festival. Let’s prepare a list of things we need to buy and estimate their prices. Is our money enough to buy them? If not, decide what should be taken out of the list. |
| Teacher initiated situation setting (2) | A class goes shopping at the supermarket and estimates the cost of their purchases (1)  
                          | Estimating the cost of things in a street market simulated in the class (1)  
                          | A street market is simulated in class. Each student is given an exact amount of money. Children estimate the cost of their purchases in order not to exceed their money. |
d. Computational estimation in school and school textbooks.

As mathematics textbooks are a predominant source in primary school in Greece, as well as in many other countries, they reflect teaching practices as well as the importance given by the curriculum to particular mathematical concepts. Findings from the questionnaire revealed how beneficial the instruction in computational estimation is considered by prospective teachers. Following the same trend, it is expected that the interviewees will also support the idea of having computational estimation as an integral part in school mathematics textbooks. Indeed, participants’ comments in the interview were aligned with their responses in the questionnaire: eight of the interviewees claimed that one or more computational estimation tasks should appear in every section of the mathematics book, if possible. This may help children develop estimation meaning and estimation strategies. Three of them also suggested that computational estimation should cover a separate section of the textbook with emphasis on its use and strategies. However, Vassilis, a 4th year student, believed that little reference to estimation is sufficient. According to him, “Mental calculations and computational estimations are important and I will try, as a future teacher, to teach them to a small extent, but they will not be my priority. You don’t start solving a mathematical problem with a computational estimation… Rules matter and are listed first in priority”. He continues “mental calculations are a matter of experience” and that “children will start thinking if this [the estimation] is right or not”, implying that the approximate result of an estimation might be confusing for younger children. This is why he comments that mental calculations and estimations should be taught to older children, implying that learning to estimate is a process that goes beyond primary school. On the other hand, Anna, a 3rd year student, overemphasized the significance of computational estimation and its early teaching suggesting that approximate answers should precede exact results. She comments “certainly exact calculations should be developed, but not necessarily in primary school”. Her comments should not be overstated, though, because she was not able to distinguish computational estimations from mental calculations at the beginning of the interview.

Interestingly, no participant in interviews revealed a sense of personal responsibility for teaching estimation to their future students or for assigning problems that elicit this type of estimation. Despite occurrences among prospective teachers of inconsistent suggestions for computational estimation tasks and their statements, analyses of interviews suggested that prospective teachers had potential for incorporating computational estimation in their future teaching, as they gained experience and training. This perceived potential was grounded in statements that indicated that they were aware of shortcomings in their suggested computational estimation tasks.
4. DISCUSSION

This study revealed two main issues. First, prospective primary school teachers believe in the importance of computational estimation for both daily life and school mathematics and appreciate its practical utility and its relation to number sense and number sense components. Its integration to the mathematics curriculum was valued highly as it will help children understand the power of computational estimation and its use in making decisions mathematically. They relate computational estimation to an adaptive and flexible way of thinking which is taught within the school environment as part of school mathematics and releases children from algorithms and rule-based methods. This new view of communicating and reasoning mathematically that encompasses the approximate and the accurate answers and gives rise of children’s informal methods when estimating was highlighted by the majority of the participants. Although this finding is in agreement with Tsao’s (2013) results with pre-service teachers, it is not in accordance with Alajmi’s study (2009) in which in-service teachers do not consider the significance and benefits of computation estimation. There were though a few participants (9%) who stressed the importance of an exact answer, confirming Alajmi’s (2009) finding that teachers consider computational estimation ‘a confusing topic’ (p.276). However, these did not reach the high percentages found in Alajmi’s study.

The second finding is related to the tasks that prospective teachers proposed for teaching computational estimation and the context in which these are used. Not all participants proposed tasks, maybe due to their difficulty to relate computational estimation to a task. From those who did propose, however, more than 55% referred to computational estimation tasks. The fact that about 28% of the tasks proposed dealt with other types of estimation, such as measurement estimation, reveals their difficulty to distinguish computational estimation from other types of estimation. Further training, thus, needs to be done in the domain of number sense.

Participants also engaged context in their computational tasks in order to show the logic behind computational estimation. They associated computational estimation with everyday experiences and activities as well as cultural practices. This finding, which is in compliance with Wager’s (2012) suggestions for meaningful ways to teach mathematics, reveals future teachers’ intention of relating children’s experiences to mathematics.

It is very promising that pre-service teachers who participated in this study acknowledged computational estimation and seem to be willing to teach it. Their clear preference in context tasks could imply a future teaching practice which incorporates out-of-school context and cultural practices. The results suggest that an embedded cultural mathematical practice or a teacher initiated situation setting, such as a simulation or a field trip, can be implemented in school practice. Fortunately, future teachers were able to identify situations most appropriate for the instruction of estimation taking into consideration their students’ informal strategies. Thus, it seems likely that they will pass on the value and the positive aspects of
computational estimation to their future students. It is very interesting, though, to further examine whether they indeed implemented computational estimation in their teaching and how they achieved it.

REFERENCES

McIntosh, A. (2004). Where we are today. In A. McIntosh & L. Sparrow (Eds.), Beyond written computation (pp.3-14). Perth: MASTEC.


**BRIEF BIOGRAPHIES**

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FACTORS CONTRIBUTING TO COMPUTATIONAL ESTIMATION ABILITY OF PRESERVICE PRIMARY SCHOOL TEACHERS

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ABSTRACT
This study focuses on the investigation of the factors that are related and contribute to computational estimation ability. We interviewed 69 students of the Department of Primary Education of the University of Crete. Moreover, these students had filled in a test. According to the analysis of the results, the factors that contribute to success at computational estimation are:
- good mathematical background and mainly good performance at exact mental computation and proportion problems,
- preference to mathematics at school,
- positive self-concept of computational estimation ability,
- positive self-concept of acquiring exact mental computation ability from the first grades of primary school,
- positive self-concept of memory ability and particularly numerical data memory ability.

Keywords: computational estimation, preservice teachers, mathematical background, affective factors.

1. INTRODUCTION
Computational estimation can play a vital role in mathematical education because it contributes to better understanding, learning and applying the algorithms (Kourkoulos and Tzanakis, 2000), as well as better understanding of numbers and their properties (Sowder, 1992). Montague and Garderer (2003) refer that computational estimation ability is associated with the acquisition of number sense1.

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1 McIntosh et.al (1997) describe number sense as “a person’s general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful and efficient strategies for managing numerical situations” (p. 3).
while Mildenhall (2011) believes that computational estimation is an integral component of number sense.

Computational estimation can be mentally, before applying the algorithm of an operation. Moreover its results can be used during the application of the algorithm in order to check the correctness of the process that is followed. Besides, they can be used after applying the algorithm in order to check the correctness of the produced result (Kourkoulos and Tzanakis, 2000). In case of the existence of an error, the application of self-correction activities is the next step. In case that a student applies such activities, he tries to find the error and correct it. However, self-correction is a basic metacognitive skill that students must develop through systematical teaching, because the acquisition of metacognitive skills: a) enables students for self-regulated learning and b) cultivates students’ positive attitude to knowledge (Matsaggouras, 2000).

Additionally, computational estimation can be used for checking the results that are produced by a calculator or a computer (Kourkoulos and Tzanakis, 2000), because there is often wrong insert of data that leads to wrong results.

Moreover, computational estimation can substitute the algorithm itself in everyday life situations where there is not the necessity of the exact result but estimation is enough. In these cases, the estimation of the result has the advantage that it usually produces faster results than the use of paper and pencil or even the calculator, when there is the appropriate exercise (Kourkoulos and Tzanakis, 2000). It is significant that it does not demand material support and consequently it can be easily applied in cases that we don’t have paper and pencil or calculator, such as when we want to estimate the cost of various products in a super market and make the decision if the money we have are enough to buy them (Lemonidis, 2013).


2. FACTORS THAT ARE RELATED TO COMPUTATIONAL ESTIMATION

The existing research on the factors related to computational estimation ability is limited; nevertheless this research identifies some such factors that can be classified in two categories: a) cognitive factors and b) affective factors. Findings of the existing research concerning these two categories are discussed below (sections 2.1 & 2.2).

2.1 Relation between computational estimation and other cognitive factors

The cognitive factors related to computational estimation ability that are mentioned in existing research studies can be classified in two groups: (i) specific cognitive factors and (ii) general cognitive factors, and more specifically general mathematical ability.
i) With regard to the first group:

- Ability to work with powers of ten is one factor that is related to computational estimation ability (Sowder and Wheeler, 1989, Rubinstein, 1985). Kourkoulos and Tzanakis (2000) mention that a prerequisite for the application of the estimation and checking criteria that rely on mental approximate computation is ability to compute mentally the results of operations with numbers that are powers of ten and with numbers of the format nx10^m (where meZ and n is of one digit, or at a more advanced level n is of two digits), especially multiplications and divisions (e.g. 700x8000=; 700x0,02x0.003=;). However, only Rubinstein presents experimental data according to which operating with tens contributed the most to the prediction of estimation performance according to regression analysis.²


- Ability to compare numbers by size is another factor that is related to computational estimation ability (Sowder and Wheeler, 1989, Kourkoulos and Tzanakis, 2000, Rubinstein, 1985).

- Other concepts and skills mentioned to be related to computational estimation ability are: a) knowledge of basic facts, b) knowledge of properties of operations and their appropriate use, c) recognition that modifying numbers can change outcome of computation (Sowder and Wheeler, 1989) and d) problem difficulty level (Dowker, 1997). However, research on examining the relation between these factors and computational estimation is very limited.

Regarding all the above studies we remark that although they indicate some specific cognitive factors which are related to computational estimation ability, most of them don’t present any experimental data and rely on theoretical analysis. Besides, there is no study (except for Rubinstein, 1985) that examines the correlation between computational estimation ability to other factors according to statistical analysis.

- Moreover, there are other specific factors that may be related to computational estimation ability but this relation hasn’t been investigated so far by researchers. Algorithmic performance of arithmetic operations could be one related factor. One reason that could justify this relation is that computational estimation can be used before the performance of an arithmetic operation and its result can be used in order to check the correctness of the result that is produced after the algorithmic performance of the operation. Besides, applying computational estimation strategies involves deep understanding of arithmetic operations.

² Rubinstein’s experimental data concern eighth graders. According to regression analysis operating with tens accounted for 42% of the variance in the estimation test score. The other mathematical skills that contributed to the prediction of estimation performance is making comparisons and getting to know the problem. These three factors altogether accounted for 46% of the variance in the estimation test score.
Ability to solve proportional problems could be another factor that is related to computational estimation ability. Its relation to computational estimation (of multiplications and divisions) is possibly due to the fact that they both belong to multiplicative structures. Besides, understanding proportion problems may contribute to computational estimation ability. For example, during the estimation of the result of a multiplication by rounding, compensation can take place in order to find a better approximation where proportional reasoning is being used: it is counted what about percentage has been cut by the one or the other one factor of the operation or has been given to them so as the result to be compensated.

Finally, ability to solve additive problems could be another factor related to computational estimation ability. This relation could be justified by the fact that compensation, which takes place in order to achieve a better approximation, involves various estimation strategies which involve posing and solving additive problems. Besides, during solving an additive problem there is the need to estimate the answer of the problem in order to check the reasonability of it.

ii) With regard to the second group (general cognitive factors):

Hogan and Brezinsky (2003) concluded that computational estimation may be subsumed under more general, well-established mathematical abilities, specifically a combination of number facility and quantitative reasoning which they call general mathematical ability. Computational estimation is not a unique ability but it is a part of this general mathematical ability. Levine (1982) found a positive correlation between math ability and estimation skill. Cilingir and Turnuklu (2009) found that students with high mathematical level and high mathematics grades are better at computational estimation. However, Gliner (1991) tried to determine the factors that contribute to computational estimation ability of preservice elementary teachers and found that average mathematics grades were negatively correlated to computational estimation performance. The fact that Gliner’s findings are not in line with other research findings points out that the relationship between computational estimation ability and general mathematical ability needs clarification. In fact, on this issue basic questions still remain without sufficient answer, such as “which are the adequate indicators of general mathematical ability which can be used so as to allow the efficient investigation of the aforementioned relationship?”

Concerning the existing research on cognitive factors related to computational estimation ability, overall we can remark that there are factors not yet or very little considered; moreover regarding factors that have received some notable attention there is an important lack of empirical investigation, in particular concerning their strength of association, as well as, their combined relation to computational estimation ability. Existing research informs us that the subject is a complex one and important research work remains to be done for elaborating a more complete understanding concerning which cognitive factors are related to computational estimation ability and what the strength and the way of their association are.
2.2. Affective factors related to computational estimation

There are few studies that investigate the affective factors which are related to computational estimation ability. In these studies, five basic affective factors have aroused that seem to be related to estimation and characterize those who have computational estimation ability: a) positive attitude to computational estimation process, b) positive self-concept of computational estimation ability, c) positive self-concept of mathematical ability d) preference to mathematics and e) tolerance for error.

- Positive attitude to computational estimation is a factor that is related to computational estimation ability and characterizes good estimators according to researchers’ positions (Sowder, 1992, Sowder and Wheeler, 1989) and findings (Bestgen et al., 1982, Reys et al, 1982, LeFevre, 1993). However, Boz and Bulut (2012) found that seventh grade students’ perceptions in the practicality of computational estimation are not related to computational estimation ability, as some good estimators believe that estimation has practicality in daily life but not in mathematics. In a study with English 12- to 14-year-olds, Morgan (1988) as well found that most of the children she interviewed did not have a clear conception of the purpose or the nature of estimation.

- Another factor that is related to computational estimation ability is positive self-concept of estimation ability (Reys et al, 1982, LeFevre, 1993). However, in a study of preservice elementary teachers, Gliner (1991) found that there is no correlation between computational estimation and self-perception of their own estimation ability.

- Positive self-concept of mathematical ability is a characteristic of good estimators (LeFevre, 1993, Sowder, 1989, Gliner, 1991). However, Boz and Bulut (2012) found that confidence in ability to do mathematics is a component of good estimators but not a distinctive factor.

- Preference to mathematics may be another factor related to computational estimation ability. However, this relation is very little investigated by existing research. Gliner (1991) examined this relation and found a positive correlation between these two factors.

- Tolerance of error is another factor that is related to computational estimation ability. Reys et al. (1980) found that tolerance for error is a characteristic of good estimators because it depends on the understanding of an estimate. Wyatt (1986) supports that good estimators are much less concerned about precision than poor estimators. Boz and Bulut (2012) found that good estimators had high tolerance for error.

Regarding these studies we can remark that relation between computational estimation and affective components has received very little attention and there are few quantitative experimental data regarding this relation. Moreover, although some
affective factors related to computational estimation have been determined, there isn’t a consensus of findings about their relation to computational estimation. Besides, other factors that are possibly related to computational estimation such as self-concept of mental computation ability, self-concept of memory ability, ambitions etc. have not been investigated.

Taking into account the above ascertainments, we designed and implemented an empirical investigation for contributing to the research concerning the factors that are related and contribute to computational estimation ability.

Our empirical investigation, which concerns preservice primary school teachers, aims to provide elements of answer to the following research questions:

- Which cognitive and affective factors are related to computational estimation ability?
- Which is the correlation between these factors and computational estimation ability?
- How groups of such factors are related to computational estimation ability?

3. METHODOLOGY

3.1 Participants

In order to achieve the purpose of the investigation and provide elements of answers to the research questions, 69 preservice elementary teachers, students of the 2nd semester of the Primary Education Department of the University of Crete (Greece), were interviewed. These students were selected from 244 students who had filled in a test during the attendance of a course of mathematical content. The students who participated to the interview were selected randomly, according to their performance at the test:

a) they had good performance\(^3\) at computational estimation (14 students)
b) they had moderate performance at computational estimation (20 students)
c) they had low performance at computational estimation (35 students)

3.2 Materials and procedure

3.2.1 Test

The test that was developed consisted of items concerning both computational estimation and other mathematical areas, from which we selected typical items that represent them. The items of other mathematical areas were included in the test in

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\(^3\) Good performance was considered the success at least at 14 of the 18 items of computational estimation and at least at 3 of the 6 items of estimating multiplications of integrals and at least at 2 of the 4 items of the rest areas of computational estimation (multiplications of decimals, divisions of integrals and divisions of decimals). Moderate performance was considered the success at 9-13 items of computational estimation. Law performance was considered the success at 0-8 items of computational estimation. We selected students of each category randomly at equal percentage of the corresponding totals (we selected randomly 14 of 49 students who had good performance, 20 of 71 students who had moderate performance and 35 of 124 students who had low performance).
order to investigate whether there is any relation between computational estimation and other mathematical skills. We selected items of other mathematical areas for which we have reasons to believe that are related to computational estimation (see section 2.1).

Specifically the test consisted of:

a) computational estimation
b) exact mental computation with numbers that are power of 10 and with numbers of the format nx10^m, where n ∈ N*, 1<n<100 and m ∈ Z
c) algorithmic performance of arithmetic operations
d) additive and multiplicative structure problems

The description of the items of the test is the following one:

a) Computational estimation

i) Find mentally between which numbers the results of the following operations are and put a cross in the appropriate small square:

<table>
<thead>
<tr>
<th>Multiplications</th>
<th>Divisions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integrals</td>
<td></td>
</tr>
<tr>
<td>57•83</td>
<td>3745•5321</td>
</tr>
<tr>
<td>289•574</td>
<td>17531:423</td>
</tr>
<tr>
<td>6143:28</td>
<td></td>
</tr>
<tr>
<td>Decimals</td>
<td></td>
</tr>
<tr>
<td>9,24•0,27</td>
<td>0,092•0,035</td>
</tr>
<tr>
<td>4,37:235</td>
<td>0,035:48,32</td>
</tr>
</tbody>
</table>

A solved example was presented to the participants:

13 • 11 10 < □ < 100 < ⊕ < 1.000 < □ < 10.000 < □ < 100.000 < □ < 1.000.000 < □ < 10.000.000 < □ < 100.000.000 < □ < 1.000.000.000

ii) Compute mentally how much about the result of the following operations is (e.g. 23•11 about: 200):

<table>
<thead>
<tr>
<th>Multiplications</th>
<th>Divisions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integrals</td>
<td></td>
</tr>
<tr>
<td>62•73</td>
<td>4832•876</td>
</tr>
<tr>
<td>2181•1485</td>
<td>67952:317</td>
</tr>
<tr>
<td>26472:43</td>
<td></td>
</tr>
<tr>
<td>Decimals</td>
<td></td>
</tr>
<tr>
<td>8,32•0,26</td>
<td>0,45•0,071</td>
</tr>
<tr>
<td>3,4:512</td>
<td>0,26:17,12</td>
</tr>
</tbody>
</table>

b) Exact mental computation with numbers that are powers of 10 and with numbers of the format nx10^m, where n ∈ N*, 1<n<100 and m ∈ Z

<table>
<thead>
<tr>
<th>Multiplications</th>
<th>Divisions</th>
</tr>
</thead>
<tbody>
<tr>
<td>with positive</td>
<td>with positive</td>
</tr>
<tr>
<td>powers of ten</td>
<td>powers of ten</td>
</tr>
<tr>
<td>3,85•10000</td>
<td>1596,76:10000</td>
</tr>
<tr>
<td>0,0467•10000</td>
<td>5,37:100</td>
</tr>
<tr>
<td>with negative</td>
<td>with negative</td>
</tr>
<tr>
<td>powers of ten</td>
<td>powers of ten</td>
</tr>
<tr>
<td>600•700</td>
<td>4900:70</td>
</tr>
<tr>
<td>90•80000</td>
<td>320000:800</td>
</tr>
</tbody>
</table>
3.2.2. Interview

The questions that the interview contained were put in order to let us determine the factors that are related to computational estimation ability. The questions can be classified in the following categories:

- Social characteristics
- Characteristics concerning participants’ knowledge background
- Attitudes and preferences
- Characteristics concerning participants’ self-concept
- Ambitions

More specifically the questions that the interview contained are:

**Social characteristics**

- Gender
- Parents’ knowledge level
- Place of origin

**Participants’ knowledge level (background)**

- Direction at secondary school
- Mathematics grades at secondary school (10th, 11th and 12th grade)
- Physics grades at secondary school (10th, 11th and 12th grade)
- Grade point average at 11th grade
- Grade point average at 12th grade
- Mathematics grade at university
- Physics grade at university
- Knowledge of foreign languages and at what level

Self-concept^{4} of:
- Mathematical ability
  \( \text{Do you believe that you are good at mathematics?} \)
  \( \text{At which courses do you believe that you are good?} \)
- Exact mental computation ability and age at which this ability was acquired
  \( \text{Can you easily compute mentally (e.g. } 8\times36 \quad 12\times15)\)?
  \( \text{If you can, from what age about have you been able to compute mentally?} \)
- Computational estimation ability
  \( \text{Do you believe that you are good at computational estimation (e.g. how much is about } 289\times324)\);
- Memory ability
  \( \text{Have you got good memory?} \)
  \( \text{Do you remember many telephone numbers by heart?} \)

Questions concerning participants’^{5} attitudes and preferences
- - Attitude to the meaning of computational estimation and reasoning this attitude
  \( \text{Do you believe that computational estimation is important?} \)
  \( \text{Very important \quad \square \quad \text{Little important \quad \square}} \)
  \( \text{Quite important \quad \square \quad \text{Not important \quad \square}} \)
  \( \text{If it is, why is it important?} \)
- - Attitude to checking operations results and reasoning this attitude
- - Courses that they liked more at school
- - Courses they like more at university
- - Liking mathematics
  \( \text{Do you like Mathematics?} \)

Questions concerning participants’ ambitions
  \( \text{What are you going to do after graduating the university?} \)
  \( \text{The interview was personal and lasted about 30-35 minutes.} \)

^{4} \text{Self-concept is the convictions and attitudes that an individual forms about himself, which involve various self-evaluations and behavioral tendencies (Burns, 1982).}

^{5} \text{Preference belongs to the field of attitudes as it involves the meaning of liking – sentimental dimension of attitude – and the meaning of behavioral tendency or intention – behavioral dimension of attitude (see Al-Khaldi and Al-Jabri, 1998).}
Correlation tests were used in order to determine the factors that are related to success at computational estimation and the strength of this correlation. Besides, stepwise\textsuperscript{6} regression analysis was used in order to determine the factors that possibly contribute and can predict success at computational estimation. This method lets us predict the dependant variable -success at computational estimation - based on a combination of independent variables -factors that are related significantly to the dependent variable (Dafermos, 2005, see also footnote 6). This analysis informs us about the way in which the related factors altogether function and permit prediction of success at computational estimation.

The score at computational estimation\textsuperscript{7} is the dependant variable (range 0-18) and all the variables concerning the test and the interview that are significantly correlated to the dependant variable (according to Pearson r) are the independent variables. The variables that entered into the model are in table 1:

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Variables & R & R \textsuperscript{2} & Adjusted R \textsuperscript{2} & R \textsuperscript{2} change and F change & p & Durbin-Watson \\
\hline
1 & 0.599 & 0.359 & 0.35 & 0.359 & 37.54 & <0.001 \\
2 & 0.717 & 0.514 & 0.499 & 0.155 & 21.075 & <0.001 \\
3 & 0.793 & 0.63 & 0.612 & 0.115 & 20.237 & <0.001 \\
4 & 0.824 & 0.679 & 0.659 & 0.05 & 9.913 & 0.002 \\
5 & 0.851 & 0.724 & 0.702 & 0.045 & 10.224 & 0.002 \\
6 & 0.866 & 0.749 & 0.725 & 0.025 & 6.226 & 0.015 \\
\hline
\end{tabular}
\caption{Stepwise multiple regression analysis to predict computational estimation score}
\end{table}

1. Exact mental computation
2. They mention preference to the course of mathematics at school
3. Self-concept of computational estimation ability
4. Proportion problems
5. Self-concept of mental computation ability having been acquired from the first grades of primary school
6. Self-concept of telephone numbers memory\textsuperscript{8}

\textsuperscript{6} On each step an independent variable that increases R-Squared the most enters into the model, conditioning that this increase is statistically significant at significant level less than 0.05. Then, the next variable that increases significantly R-Squared the most enters. Then, the inserted variables are checked to see if one of them satisfies the removal criterion. That is the variable that increases R-Squared the least is removed, conditioning that this increase is non-significant at significant level 0.01. Then the variable that increases R-Squared the most enters into the model, conditioning that this increase is statistically significant at significant level less than 0.05 and all the inserted variables are checked to see if they satisfy the removal criterion, e.t.c. Therefore, a variable enters into the model if it is recognized to be a significant predicting factor and it is removed if once it stops to be a significant predicting factor. Given that the significant level is less for the entrance of a variable than the significance level for the removal of a variable, the procedure does not get into an infinite loop (Dafermos, 2005).

\textsuperscript{7} The success at each item of computational estimation accounts for one point (see the description of the test).

\textsuperscript{8} The statistical variables (1) “Exact mental computation” and (4) “Proportion problems” are the students’ scores at the corresponding areas of the test (see the description of the test). The success at each item of an area
In table 2 there is the correlation between the variables which entered into the model and the dependent variable:

**Table 2:** Correlation between the variables which entered into the stepwise multiple regression model and the computational estimation score

<table>
<thead>
<tr>
<th>Variables</th>
<th>Computational estimation score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact mental computation</td>
<td>0,599</td>
</tr>
<tr>
<td>They mention preference to the course of mathematics at school</td>
<td>0,565</td>
</tr>
<tr>
<td>Self-concept of computational estimation ability</td>
<td>0,517</td>
</tr>
<tr>
<td>Proportion problems</td>
<td>0,587</td>
</tr>
<tr>
<td>Self-concept of mental computation ability having been acquired from the first grades of primary school</td>
<td>0,395</td>
</tr>
<tr>
<td>Self-concept of telephone numbers memory</td>
<td>0,251</td>
</tr>
</tbody>
</table>

However, we should note that there are variables which don’t enter into the model although they are significantly correlated to the dependent variable. This is owing to the fact that they are significantly correlated to variables that have already entered into the model and the main part of their predictability is in the already inserted variables. Therefore, they cannot enter because there would be the phenomenon of multicollinearity (Dafermos, 2005). These variables<sup>9</sup> are shown in table 3:

**Table 3:** Correlation between the variables that don’t enter into the stepwise multiple regression model and computational estimation score under the control of the variables that have entered into the model (partial correlation)

<table>
<thead>
<tr>
<th>Category of questions</th>
<th>Variables (see footnote 9)</th>
<th>Responses</th>
<th>Correlation to computational estimation score (Pearson r)</th>
<th>Correlation to estimation score under the control of entered variables (Pearson r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Social characteristics</td>
<td>Gender</td>
<td>Boy</td>
<td>0,331*</td>
<td>0,119</td>
</tr>
</tbody>
</table>

accounts for one point at the corresponding student’s score. The variables (2), (3), (5), (6) are statistical variables with two values (1, 0). For each variable, the value 1 is attributed to a student if his/her answer is affirmative to the corresponding question; for all other answers the value 0 is attributed to the student.

<sup>9</sup> The statistical variables “Additive problems”, “Subtractions” and “Divisions” are the students’ scores at the corresponding areas of the test (see the description of the test). The success at each item of an area accounts for one point at the corresponding student’s score. All the other statistical variables of this table have two values (1, 0). For each variable, the value 1 is attributed to a student if his/her answer corresponds to what is described in the “response” column of the table; for all the other answers the value 0 is attributed to the student.
## Category of questions | Variables (see footnote 9) | Responses | Correlation to computational estimation score (Pearson r) | Correlation to estimation score under the control of entered variables (Pearson r)
--- | --- | --- | --- | ---
### Background
Additive problems | Score at each of these areas | 0.262* | 0.088
Subtractions | 0.285* | -0.147
Divisions | 0.44** | 0.012
Direction at school | Sciences\textsuperscript{10}/Technological | 0.373* | -0.083
Grade point average at 11\textsuperscript{th} grade | ≥18,5 | 0.237* | 0.047
Grade point average at 12\textsuperscript{th} grade | ≥18,5 | 0.285* | -0.021
Math grades at school | 19-20 | 0.488** | -0.07
Physics grades at school | 19-20 | 0.35* | -0.141
### Attitudes and preferences
They mention preference to the course of math at university | Yes | 0.495** | 0.05
They mention preference to the course of physics at school | Yes | 0.39** | 0.028
They mention preference to the course of physics at university | Yes | 0.267* | -0.157
If they like mathematics | Yes | 0.543** | 0.113
If they believe that computational estimation is important and reasons for that | Yes and they mention more than one reason for that | 0.385** | 0.188
### Self-concept
They mention that they believe that a course at which they are good is mathematics | Yes | 0.567** | 0.083

\textsuperscript{10} In Greece both the direction of Sciences and the Technological direction involve the course of Mathematics while Theoretical direction does not involve this course.
**FACTORS CONTRIBUTING TO COMPUTATIONAL ESTIMATION ABILITY OF PRESERVICE PRIMARY SCHOOL TEACHERS**

<table>
<thead>
<tr>
<th>Category of questions</th>
<th>Variables (see footnote 9)</th>
<th>Responses</th>
<th>Correlation to computational estimation score (Pearson r)</th>
<th>Correlation to estimation score under the control of entered variables (Pearson r)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>They mention that they believe that a course at which they are good is physics</td>
<td>Yes</td>
<td>0.382**</td>
<td>0.075</td>
</tr>
<tr>
<td></td>
<td>If they believe that they are good at mathematics</td>
<td>Yes</td>
<td>0.534**</td>
<td>-0.044</td>
</tr>
<tr>
<td></td>
<td>If they believe that they are fluent at mental computation</td>
<td>Yes</td>
<td>0.449**</td>
<td>0.05</td>
</tr>
</tbody>
</table>

*Correlation is statistically significant at significant level less than 0.05.*  
**Correlation is statistically significant at significant level less than 0.001.*

We remark that the correlation between these variables and computational estimation score is statistically significant (column 4). However, the correlation between these variables and computational estimation score isn’t statistically significant any more under the control of the variables that have entered into the regression model (column 5), because of their correlation to these variables. This is the reason why these variables do not enter into the model, as their prediction potential is covered by the variables that have already entered into the model.

The variables that entered into the regression model concern the following categories of characteristics (see graph 1 and table 1):

- Mathematical background – cognitive factors which account for 40.9% of R-Squared (exact mental computation accounts for 35.9% of R-Squared, proportion problems account for 5% of R-Squared).
- Preferences (preference to the course of mathematics at school) which account for 15.5% of R-Squared
- Self-concept (accounts for 18.5% of R-Squared) of:
  - computational estimation ability which accounts for 11.5% of R-Squared
  - exact computation ability having been acquired from the first grades of primary school which accounts for 4.5% of R-Squared
  - memory ability (telephone numbers memory) which accounts for 2.5% of R-Squared.
Graph 1: Factors that compose R-Squared according to stepwise multiple regression analysis

* Variance that is not predicted by factors that have been included in the model.

In the beginning, cognitive factors which concern mathematical background seem to play a vital role for the success at computational estimation and especially exact mental computation (prerequisite for computational estimation by applying rounding strategies) and proportion problems.

Besides, their self-concept of computational estimation ability and early acquisition of exact computation ability seem to be very important. In addition to these, there is evidence for the important role of participants’ positive self-concept of numerical data memory ability.

Also, participants’ preference to mathematics seems to be very significant.

All these factors account for 74.9% of the total variance in computational estimation score, which is a high percentage for the field of Social Sciences.

5. DISCUSSION

Data analysis indicated a number of factors that are related to computational estimation ability (tables 2, 3). Besides, it indicated the strength of the correlation of these factors to success at computational estimation (tables 2, 3). In addition, some of them can be predictors for success at computational estimation (table 1, graph 1). Therefore, the analysis determined the combined way in which some factors interact and predict success at computational estimation. So, this study, providing elements to the research questions, lights aspects of computational estimation ability that had not been investigated. The factors that seem to be very important for computational estimation ability are the following ones:

11 The graph shows the R-Squared change that the factors which enter into the model cause, according to table 1.
a) Background

i) Specific cognitive factors of mathematical background

The first cognitive factor that is significantly related and contributes to success at computational estimation is exact mental computation with powers of ten and with numbers of the format nx10^m, where n ∈ N*, 1<n<100 and m ∈ Z (table 1, 2) which is prerequisite for computational estimation when using rounding strategies (Kourkoulos and Tzanakis, 2000). This result is consistent with the findings and the positions of other studies (Sowder and Wheeler, 1989, Rubinstein, 1985, Kourkoulos and Tzanakis, 2000) but in the present study this factor is not only significantly correlated to success at computational estimation (table 2) but it is a major factor that contributes to it (table 1).

The second significant cognitive factor is ability to solve proportion problems. The statistical variable that concerns students’ ability to solve proportion problems has important correlation to their computational estimation score (table 2). Moreover, it is a variable that contributes to prediction of success at computational estimation (table 1). These findings are interesting because they point out that there is an important relation between the ability to solve proportion problems and computational estimation ability. However, this relation hadn’t been investigated before. Nevertheless, our investigation is a first investigation on this issue, and further research is needed to obtain a deeper and more complete understanding of this important relation.

Besides, there is a significant correlation between computational estimation score and scores concerning other mathematical areas and specifically divisions, subtractions and additive problems which they don’t enter into the model because of their relation to other variables that have already entered into the model (table 3). There is no previous research concerning the relations between students’ performances to the above mathematical areas and to computational estimation.

Taking into account the significant relations between the aforementioned factors and success at computational estimation, we can suppose that mathematical background plays a vital role for computational estimation ability.

ii) General background

In the present study we used math grades at school as an indicator of general mathematical ability and we remarked that there is an important correlation between this indicator and computational estimation score (table 3). This result is in line with previous research findings (Levine, 1982, Hogan and Brezinsky, 2003, Cilingir and Turnuklu, 2009) but contrasts the result of Gliner’s study (1991) who found that preservice elementary teachers’ average mathematics grades were negatively correlated to computational estimation performance.

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12 A possible explanation for the relation between the ability to solve proportion problems and computational estimation is presented in section 2.1.
Moreover we used grade point average at 11th and 12th grade, physics grades at school and direction at school as indicators of general background. All these indicators have a significant correlation to computational estimation score (table 3). However, they don’t enter into the model because their prediction potential is contained in factors that have entered into the model. This information is interesting as these factors and their relation to computation estimation hadn’t been investigated before.

Nevertheless, further research is needed for elaborating a deeper and more complete understanding concerning the way in which these factors are related to computational estimation ability. Moreover, so far research on the subject had not investigated factors of general background, besides general mathematical background; our results indicate that interesting and fruitful research work remains to be done concerning this area.

**b) Preferences**
Participants’ preference to the course of mathematics at school has an important relation to success at computational estimation (table 2) and it is a major factor that contributes to it (table 1). Besides, liking mathematics, preference to the course of mathematics at university, preference to the course of physics at school and preference to the course of physics at university are factors which are significantly related to success at computational estimation (table 3). These factors don’t enter into the regression model because they are significantly correlated with other variables that have already entered the model. These findings point out that there is an important relation between preference to mathematics and physics and computational estimation ability. This relation has been investigated very little by previous researches. Our findings are in line with the results of Gliner’s study (1991). However, in the present study this relation arouses from more aspects using more indicators and it is more intensive.

Of course, it is possible that the correlation of preference to mathematics to success on computational estimation is owing to the two-sided relation between preference to mathematics and performance at mathematics. Students who like mathematics are interested in doing mathematics and consequently they have possibly better performance. Besides, students who have good performance at mathematics acquire positive attitude to it.

**c) Self-concept**
Participants’ self-concept seems to play an important role for success at computational estimation. Specifically:

1) **Self-concept of computational estimation ability**
Self-concept of computational estimation ability seems to be very important for success at computational estimation (tables 1, 2), supporting previous research findings (Reys et.al, 1982, LeFevre, 1993) but contrasting Gliner (1991) who found that there is no correlation between computational estimation and self-
perception of estimation ability. So, according to the present study, it seems that participants’ image of their computational estimation ability complies with their real image.

**ii) Self-concept of exact mental computation ability and the age of acquiring this ability**

Participants’ self-concept of the age of acquiring exact mental computation ability (from the first grades of primary school) is an important factor for predicting success at computational estimation (tables 1, 2) that hadn’t been investigated before. Besides, self-concept of mental computation ability is a factor that has an important correlation to success at computational estimation (table 3) that hadn’t been investigated before too.

Therefore, if the image that participants have for themselves with regard to exact mental computation ability and the age of acquiring this ability complies with their real image, there is evidence that early exercise on mental computation and acquisition of this ability may contribute to the development of computational estimation ability.

**iii) Self-concept of memory ability**

Students’ self concept of memory ability and especially of numerical data memory ability emerges to be an important factor for success at computational estimation (tables 1, 2) that hadn’t been investigated before. The significance of memory on computational estimation ability possibly relies on the fact that estimating the result of an operation by using rounding strategies demands: a) reformulation of numerical data, b) constraining the new numerical data in short-term memory and c) mental computation of them. Therefore, short-term memory is possibly very important for the process of estimation. Moreover, it is necessary to recall the process of estimation from long-term memory. However, further research is necessary in order to extract safer conclusions.

Data analysis gives evidence that computational estimation ability is a multifactorial phenomenon. The factors that contribute significantly for predicting success at computational estimation concern: a) mathematical background, b) preference to mathematics and c) self-concept.

So, according to the results of the study, there is evidence that a part of computational estimation ability is related to students’ knowledge (mathematical background) and another part of this ability is related to students’ attitude (self-concept, preferences).

However, the multiple regression generated in this study accounted, at best, for 74,9% of the variance in computational estimation performance. Further research is needed: a) to confirm the findings of the present research, b) to investigate other factors that possibly account for the remainder such as training in mental calculation at home or at school, teaching methods and approaches having been used at school, parents’ job, parents’ computational ability or parents’ general mathematical ability.
and c) to investigate the reasons why these factors interact and influence computational estimation ability.

References


**BRIEF BIOGRAPHIES**

**Georgia Chalepaki** is a teacher of a primary school. She has graduated from the Department of Primary Education – National and Kapodistrian University of Athens and from the Department of Mathematics – University of Crete. She has received a master and a Ph.D. in Didactic of Mathematics from the Department of Primary Education of the University of Crete.

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PROSPECTIVE TEACHER’S EFFICIENCY AND FLEXIBILITY IN PREP AND MENTAL CALCULATION OF TWO-DIGIT MULTIPLICATIONS

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ABSTRACT

Students’ and adults’ knowledge of prep (Campbell & Xue, 2001, LeFevre, & Liu, 1997, LeFevre et al., 1996, Metcalfe & Campbell, 2008) has been studied extensively. However, a few studies have analyzed students’ strategies in mental calculations of multi digit multiplications (Baek Jae-Meen, 1998; Heirdsfield, A. et al. 1999; Murray, H., et al. 1994; Lemonidis, 2013). In addition, there are no studies analyzing adults’ strategies or examining their mental flexibility in two-digit multiplications, as well as comparing this flexibility with their knowledge of multiplication tables. The main research questions in this paper are (a) which strategies are used by prospective teachers, (b) how flexible they are in mental calculations of two-digit multiplications and (c) how this flexibility is associated with their knowledge of the multiplication tables. Their flexibility in strategies employed to multiply two-digit numbers was examined by using the method choice / no-choice by Lemaire & Siegler, (1995). The results showed that prospective teachers are not flexible in two-digit multiplications and they mostly use the written algorithms mentally in order to calculate. Moreover, their flexibility to mentally calculate two-digit multiplications is associated with their knowledge in prep and their prep response time.

Keywords: Prospective teacher’s knowledge, mental multiplication, knowledge in prep, mental multiplication of two-digit number, flexibility in multiplication.
1. INTRODUCTION

Many studies have been conducted examining the prospective primary school teachers’ knowledge in mathematics. In many of these studies it was indicated deficiency in mathematical content knowledge of prospective teachers (e.g. Ball, 1991; Ball, et al. 2008; Lampert, 1986; Ma, 1999). In this paper, a total of 50 Greek students- prospective teachers- was examined by correlating their skills in mental calculations of simple-digit numbers (multiplication tables) and two-digit multiplications.

The multiplication tables is a basic knowledge that is taught for the first time in the second and third grade of elementary school in Greece and it is considered as important and necessary knowledge, because more complex mental and written calculations with multiplications and divisions are based on it. In the recent decades, mental calculations have been a key curricular content on a global basis (e.g. in England (DfEE, 1999, DfES 2007), in the USA (NCTM, 1989, 2000), in the Netherlands, the Dutch Specimen of a National Program for Primary Mathematics (Treffers & De Moor, 1990), in Australia, the National Statement on Mathematics for Australian Schools (Australian Education Council, 1991) etc. In Greece the mental calculations emerge in the Cross Curriculum Framework (Δ.Ε.Π.Π.Σ., 2003). Students should, among the other maths’ operations, be able to perform mental two-digit multiplications. Therefore, in this paper we examined students as adults and especially as prospective teachers, who will be invited to teach these contents, as well as we examined and analyzed their content knowledge. We desided on examining two-digit multiplications because multi-digit multiplications are bound to the use of the algorithm, while two digit ones are closely related to metal mental strategies.

1.1 Research on the multiplication tables in adults

Through the years, the study on the development of prep strategies showed that procedural strategies have been gradually replaced by immediate recall from memory. This strategy appears during the early school years and develops gradually over the next years by storing in memory more and more numerical facts (Koshmider & Ashcraft, 1991; Siegler, 1988; Siegler & Shrager, 1984). When reaching tertiary education, most of the key numerical facts have probably been dealt with so often that their recall from memory is the dominant strategy (Campbell & Xue, 2001; LeFevre, & Liu, 1997; LeFevre et al., 1996).

However, studies related to adults’ solving simple arithmetic multiplication problems have shown that prior experience does not necessarily result into the exclusive strategy use of recalling numerical facts. LeFevre et al. (1996) found that adults use immediate recall in 80% of the tests, but also use repeated addition, rhythmic counting and product construction. The following three factors, identified in literature, seem to have a significant influence on the answers given by students and adults in prep:
a) **Problem-size effect.** It is easier for both students and adults to calculate products with small numbers. As the numbers of the products grow, the difficulty and the mistakes of students or adults increase (Campbell & Graham, 1985; LeFevre et al., 1996). The response time is greater for products with larger numbers (Campbell & Graham, 1985; LeFevre et al., 1996).

b) **Ties effect.** The products containing doubles (i.e. two factors are the same, e.g. 7×7) are easier for students and adults than products with corresponding numbers, which are dissimilar, e.g. 7×6.

c) **Effects of 5-operand problems.** Multiplying with one of the factors being number 5, for example 5×8, is easier for students and adults and is solved faster than other products of comparable size, as for example 6×7 (Campbell, 1994; Campbell & Graham, 1985; LeFevre et al., 1996).

Based on these data from research, we separated the 12 products of the multiplication tables for the needs of the current study into two groups: a) products with small and easy numbers, b) products with large numbers (see 2.3).

### 1.2 Strategies employed in multi-digit multiplications

Although there are several studies examining the strategies of students and adults in single-digit multiplications (prep), there has been a limited number of studies regarding the strategies that students use in two-digit or multi-digit multiplications, (Baek Jae-Meen, 1998; Heirdsfield, et al., 1999; Murray et al., 1994; Lemonidis, 2013). Based on the existing literature, Lemonidis (2013, 258-261) suggested the following classification of strategies for multi – digit multiplications:

1. **Direct Modeling.** They model the problem with objects or a drawing, and they count the total number of objects, the number of groups or the number of objects in each group.

2. **Counting Strategies.** They count the numbers of the product by using all numbering forms, by skipping forward, by repeated addition and by employing doubling strategies, e.g. 4×15: 15, 30, 45, 60, or 8×25: 2×25 = 50, 50 +50 = 100, 100 +100 = 200.

3. **Direct retrieval.** They directly recall from their memory a known multiplication numerical fact or a production of a numerical fact, e.g. 6×11 = 66, 4×12 = 48.

4. **Partitioning number strategies.** They partition one or both terms of the operation in smaller numbers, so that they can multiply them more easily.
   4.1 Partitioning a number based on its positional value. They partitioning one number based on the numerical system positional value and multiply the parts with the other number, e.g. 8×25 = 8×(20 +5) = (8×20) + (8×5) = 160 +40 = 200.
   4.2 Partitioning both numbers based on their positional value. They partition the multiplier and the multiplicand numbers based on the positional value, e.g. 11×18 = (10 +1)×(10 +8) = (10×10) + (10×8) + (1×10) + (1×8).
4.3 Partitioning a number into nondecade numbers. They partition the multiplier or the multiplicand, not based on the positional value of the numbering system, e.g. 8x25 = 2x4x25 = 2x100 = 200.

5. Holistic or compensating strategy. They regulate one or both terms of the operation, so that the calculation becomes easier, e.g. 8x99 = 8x (100 - 1) = 800-8 = 792.

Lemonidis (2013) examined the performance and the strategies used by the fourth grade primary school students in mental multiplications, same operations of multiplication are the same in the current survey. In that study (Lemonidis, 2013, p.268), students were tested before and after a teaching intervention in mental calculations. The 72.5% and 47% of the students succeed in calculating 8x25 and 8x99 respectively, while after the intervention success rates increased to 80.5% and 68% respectively. Regarding the use of strategies, students in general used the mental strategy of the written algorithm for two-digit multiplications and they did not use holistic strategy. Specifically, fourth grade students used holistic strategies in 8x25 and 8x99 operations by 0% and 14% respectively, before the intervention, while after the intervention rates increased in 32% and 36% respectively.

1.3 Flexibility in mental calculations

The method choice / no choice by Lemaire and Siegler, (1995) has been successfully implemented to assess children’s and adults' strategy choices in diverse mathematical domains, including solving one-step multiplications (Siegler & Lemaire, 1997) and one-step additions and subtractions (Torbeyns, Verschaffel, & Ghesquiere, 2004, 2005), currency conversion (Lemaire & Lecacheur, 2001), computational estimation (Lemaire & Lecacheur, 2002) and numerosity judgment (Luwel, Verschaffel, & Lemaire, 2005).

Ligouras (2012) used the method choice / no choice to examine the flexibility in mental two-digit multiplications of the sixth grade primary school students. In that study, five multiplications were used (8x25, 9x21, 12x18, 19x30, 15x49), which were the same with the present study except for 12x18 which, in the present study, was 11x18. Moreover, in the present study there was a sixth product, that of 8x99. The results of Ligouras’ research showed that the average success rate for the choice state was 30.49% while the state for no choice, where they were obliged to use the holistic strategy, success rate drops to 15.96%. The average flexibility rate of students in mental multiplication was very low and reached only 5.92%.

1.4 The present study

The main research question was whether future teachers’ knowledge of prep affects their performance and flexibility in mental two-digit multiplications. This paper also aims at answering the following questions: What is the performance, and what
strategies do prospective teachers employ when calculating products of the multiplication tables?
What is the performance, the strategies used and the flexibility in using strategies of prospective teachers concerning mental two-digit multiplications?

2. METHOD

2.1 Participants
The sample of the study consisted of 50 undergraduate students (N = 50) of the Department of Primary Education, University of Western Macedonia Florina. Forty (40) of them were female (80%) and 10 (20%) were male. Their age ranged from 18 to 22 years (M=19.94 years, SD=1.43).

2.2 Procedure
The study was conducted at the Department of Education, University of Western Macedonia, Florina in May and June 2011. Students were interviewed and the questions were presented on a computer, which also measured the response time. The time recording began with the presentation of the question and stopped when the examinee gave his/her first answer. The researcher presented the question to the student, recorded the response and the strategy employed to perform the operation, asking the student to think aloud and express the way s/he thought and observing any obvious external behavior, such as counting on fingers.

2.3. Tasks
Students were asked to solve mentally 12 prep operations and 6 operations of two-digit multiplications. The prep operations were classified into the following two groups:
1. prep with small and easy numbers: 6x6, 3x8, 5x9, 5x7, 4x9, and 7x7.
2. prep with large numbers: 8x9, 7x8, 6x7, 6x9, 9x8, 9x9

In the prep with small and easy numbers, products with small numbers of 3, 4 and 5 were included, as well as the double products of 6x6 and 7x7, which are considered easy. In the prep with large numbers included multiplications by numbers of 6, 7, 8, and 9.

The 6 operations of two-digit multiplications were classified into the following two groups:
1. Multiplication by simple-digit multiplier and two-digit multiplicand: 8x25, 9x21 and 8x99.
2. Multiplications by two-digit multiplicand and multiplier: 11x18, 19x30 and 15x49.
2.4. Choice / no choice method for flexibility

We chose to use the choice / no choice method (Lemaire & Siegler, 1995) to measure the flexibility of students in the six operations of two-digit multiplications.

Students were tested in solving 6 two-digit multiplications by taking part in three interviews. In the first interview (choice), students were asked to calculate each of the 6 multiplications using freely any convenient calculation strategy to solve the operations. It is worth mentioning that the students had as much time as they needed.

In the second interview (no choice 1), students were asked to solve the six multiplication operations, which were presented in a different order from that of the first interview, only by using the holistic strategy. If their response was not extracted by using holistic strategy, it was considered a wrong one (even if it was a correct response). In the third and last interview (no choice 2), students were asked to solve the six operations that were presented in different order, using whatever strategy they wanted except the holistic strategy. If they used the holistic strategy the answer was considered wrong.

Flexibility was measured, with the method choice/ no choice, at six operations with two-digit multiplications for those students who answered correctly. Specifically, students were characterized as flexible in the following two cases:

1. When the strategy used in the first interview was holistic, in the second interview it was also holistic and in the third interview the strategy employed was not holistic. The time they needed to give the correct response was greater in the third interview than in the second one.

2. When the strategy used in the first interview was not holistic, in the second interview it was holistic and in the third interview the strategy employed was not holistic. The time they needed to give the correct response was lower in the third interview than in the second one.

Students were characterized as no flexible in the following two cases:

1. When the strategy employed in the first interview was holistic, in the second interview it was also holistic and in the third interview the strategy used was not holistic. The time needed to give the correct response was greater in the second interview than in the third interview.

2. When the strategy employed in the first interview was not holistic, in the second interview it was holistic and in the third interview it was not holistic.

3. RESULTS

3.1. Accuracy and strategies in multiplication tables

3.1.1. Accuracy in multiplication tables

Two sets of products were presented to prospective teachers: a) products with small numbers and easy to calculate and b) products with large numbers. The performances
for each prep product, the mean time and the standard deviation of the time, as well as the various strategies used in order to find the product are presented in the Table 1.

Table 1: Effectiveness, mean response time (sec), standard deviation and strategies used for the products in the multiplication tables

<table>
<thead>
<tr>
<th>Operation</th>
<th>Effective (%)</th>
<th>Mean Time</th>
<th>Standard deviation</th>
<th>Direct retrieval</th>
<th>Derived-fact</th>
<th>Recite all multipl. table</th>
</tr>
</thead>
<tbody>
<tr>
<td>9x9</td>
<td>50 (100)</td>
<td>1.14</td>
<td>.756</td>
<td>48 (96)</td>
<td>0</td>
<td>2 (4)</td>
</tr>
<tr>
<td>7x7</td>
<td>48 (96)</td>
<td>1.88</td>
<td>2.454</td>
<td>49 (98)</td>
<td>0</td>
<td>1 (2)</td>
</tr>
<tr>
<td>6x7</td>
<td>48 (96)</td>
<td>2.06</td>
<td>2.985</td>
<td>43 (86)</td>
<td>4 (8)</td>
<td>1 (2)</td>
</tr>
<tr>
<td>5x9</td>
<td>48 (96)</td>
<td>1.44</td>
<td>1.145</td>
<td>44 (88)</td>
<td>2 (4)</td>
<td>2 (4)</td>
</tr>
<tr>
<td>6x6</td>
<td>46 (92)</td>
<td>1.46</td>
<td>1.842</td>
<td>45 (90)</td>
<td>0</td>
<td>1 (2)</td>
</tr>
<tr>
<td>5x7</td>
<td>46 (92)</td>
<td>2.70</td>
<td>4.514</td>
<td>44 (88)</td>
<td>1 (2)</td>
<td>1 (2)</td>
</tr>
<tr>
<td>4x9</td>
<td>46 (92)</td>
<td>3.16</td>
<td>5.474</td>
<td>38 (76)</td>
<td>4 (8)</td>
<td>4 (8)</td>
</tr>
<tr>
<td>9x8</td>
<td>46 (92)</td>
<td>3.08</td>
<td>4.049</td>
<td>43 (86)</td>
<td>2 (4)</td>
<td>1 (2)</td>
</tr>
<tr>
<td>8x9</td>
<td>45 (90)</td>
<td>4.46</td>
<td>8.315</td>
<td>40 (80)</td>
<td>4 (8)</td>
<td>1 (2)</td>
</tr>
<tr>
<td>3x8</td>
<td>43 (86)</td>
<td>2.24</td>
<td>2.591</td>
<td>42 (84)</td>
<td>0</td>
<td>1 (2)</td>
</tr>
<tr>
<td>7x8</td>
<td>41 (82)</td>
<td>3.42</td>
<td>5.990</td>
<td>37 (74)</td>
<td>2 (4)</td>
<td>2 (4)</td>
</tr>
<tr>
<td>6x9</td>
<td>39 (78)</td>
<td>4.34</td>
<td>5.586</td>
<td>32 (64)</td>
<td>3 (6)</td>
<td>5 (10)</td>
</tr>
</tbody>
</table>

In the 12 prep questions the average performance of students is M = 91% and SD = 0.1%. Only 40% of the total number of students answered correctly to all (12) questions, 68% answered correctly to 11 questions and 88% answered correctly to 10 questions. The 12% of the students could not answer the 3 or 4 of the twelve given products in the multiplication tables. The product 6x9 could not be answered by 22% of the students and the product 7x8 could not be solved by 18% of students.

The average success rate in the multiplications by small numbers was 92.33%, with a standard deviation of 1.08%, while the average success rate in multiplications by large numbers was 89.67%, with a standard deviation of 1.38%. No statistically significant differences were recorded between the success rate of multiplications with large and small numbers (t = 1.184, df = 49, p = .242). Therefore, for the statistical analysis that follows, the average of the success rates in small and large numbers multiplications will be used as a single variable, which represent students’ success rate in multiplications.

However, the students needed significantly more time to find the products in the multiplication tables with large numbers in relation to the products in the multiplication tables with small numbers (t = 3.438, df = 49, p < .005). Specifically, students needed an average of 2.15 seconds with a standard deviation of 2.15 seconds in order to find the products in the multiplication tables with small numbers. On the other hand, they needed on average 3.08 seconds with a standard deviation of 3.27 seconds for the products in the multiplication tables with large numbers.
3.1.2 Strategies in multiplication tables

The strategies used by the students to find the products of the multiplication tables with both small and large numbers were the following:

- **direct recall** of the product from memory, for example 8x9 they responded immediately,
- **derived –fact**, used other multiplications facts or subtractions and multiplications facts (e.g. to calculate 8x9, they calculate 8x10 = 80-8) and
- **recitation** of all multiplication table. For example, to calculate the product 6x9 they refer to the whole column of the multiplication table 9, 1x9 = 9, 2x9 = 18, 3x6 = 18, ... 6x9 = 54.

The basic strategy of calculation, which is used by the majority of students (64% to 98%) was the direct recall of the product. Very few students used the strategy of derived –fact. The greater percentage of this strategy (8%) appeared in the products: 6x7, 4x9 and 8x9. Similarly, 10% was the greatest percent regarding the strategy "recite all the multiplication table" and appeared in the calculation of the product 6x9, which seemed to be the most difficult. Based on the strategy of direct recall, the products could be classified into those with a very high percentage (from 84% to 98%) and those with a high percentage (from 64% to 82%) of direct recall: In the first category classified the products: 7x7 (98%), 9x9 (96%), 6x6 (90%), 5x7 (88%), 5x9 (88%) and 6x7 (86%), 9x8 (86%) and 3x8 (84%). In the second category classified the products: 8x9 (80%), 4x9 (76%), 7x8 (74%) and 6x9 (64%).

3.2. Flexibility and multiplication tables

In order to examine possible differences in students’ success rate in the multiplication tables according to their level of flexibility, the technique of One Way ANOVA was used with the success rate as the dependent variable and the level of flexibility as the independent variable. The analysis indicated that students’ success rate in the multiplication tables (F(2,47)= 3.60, p < .05) was significantly affected by the students’ flexibility. More specifically, in accordance with the post hoc LSD test (.05 level), students with a medium level of flexibility are performing significantly higher in the multiplication tables (M = 96.43%, SD = 5.4%) compared to students of low (M = 88.10%, SD = 10.7%) and ‘none’ flexibility level (M = 89.39%, SD = 9.70%). One Way ANOVAs were also used in order to examine possible differences in students’ time response in the case of large numbers as well as in the case of small numbers multiplication, with the level of flexibility as the between subjects variable. The analysis indicated that the level of flexibility significantly affects students’ time response in the case of large numbers multiplication (F(2,47)= 4.94, p < .05). The post hoc LSD test (.05 level) indicated that students with a medium level of flexibility needed significantly less time (M = 0.93sec, SD = 0.46sec) to find the result of multiplication tables with large numbers compared to students of low (M = 3.75sec, SD = 3.14sec) or ‘none’ flexibility level (M = 4.03 sec, SD = 3.79 sec). However, the
analysis indicated that students’ flexibility does not seem to affect the time needed for finding the result in the case of the multiplication tables with small numbers ($F(2, 47) = 3.13, p = .053$). Nevertheless, the more flexible students need less time ($M = 0.98\text{sec}, SD = 0.81\text{sec}$) to find the result of the multiplication tables with small numbers compared to students of low ($M = 2.54\text{sec}, TA = 1.45\text{sec}$) or zero ($M = 2.64\text{sec}, SD = 2.80\text{sec}$) flexibility level (Table 2).

**Table 2.** Mean time and standard deviation answer time (sec) in the multiplication tables.

<table>
<thead>
<tr>
<th>Multiplication tables Flexibility</th>
<th>Small numbers</th>
<th>Large numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean time</td>
<td>Standard Deviation</td>
</tr>
<tr>
<td>Zero</td>
<td>2.64</td>
<td>2.80</td>
</tr>
<tr>
<td>low</td>
<td>2.54</td>
<td>1.45</td>
</tr>
<tr>
<td>medium</td>
<td>0.98</td>
<td>0.81</td>
</tr>
<tr>
<td>Total</td>
<td>2.15</td>
<td>2.15</td>
</tr>
</tbody>
</table>

### 3.3. Accuracy and strategies in mental multiplication of two digit numbers

#### 3.3.1 Accuracy

**Table 3.** Performance and strategies in two-digit multiplications

<table>
<thead>
<tr>
<th>Situation Choice</th>
<th>Operation</th>
<th>Accur. (%)</th>
<th>Accur. (%)</th>
<th>Accur. (%)</th>
<th>Accur. (%)</th>
<th>Accur. (%)</th>
<th>Accur. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8x25</td>
<td>40 (80)</td>
<td>21 (42)</td>
<td>11 (22)</td>
<td>18 (36)</td>
<td>43 (86)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>9x21</td>
<td>33 (66)</td>
<td>35 (70)</td>
<td>8 (16)</td>
<td>10 (20)</td>
<td>41 (82)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8x99</td>
<td>26 (52)</td>
<td>30 (60)</td>
<td>15 (30)</td>
<td>-</td>
<td>19 (38)</td>
<td>31 (62)</td>
</tr>
<tr>
<td></td>
<td>19x30</td>
<td>31 (62)</td>
<td>21 (42)</td>
<td>10 (20)</td>
<td>16 (32)</td>
<td>25 (50)</td>
<td>37 (74)</td>
</tr>
<tr>
<td></td>
<td>11x18</td>
<td>26 (52)</td>
<td>24 (48)</td>
<td>1 (2)</td>
<td>20 (40)</td>
<td>19 (38)</td>
<td>16 (32)</td>
</tr>
<tr>
<td></td>
<td>15x49</td>
<td>12 (24)</td>
<td>14 (28)</td>
<td>10 (20)</td>
<td>10 (20)</td>
<td>17 (34)</td>
<td>15 (30)</td>
</tr>
</tbody>
</table>

As shown in Table 3 in the first column of the situation choice, students’ performance in mental two-digit multiplications ranged from 24% (15x49) to 80% (8x25).

The success rate in multiplying a two-digit multiplier and single-digit multiplicand ($M = 66\%, SD = 30\%$) is significantly higher than the success rate in multiplying two-digit multipliers and two-digit multiplicands ($M = 44\%, SD = 30\%$). This difference in student success rates in both cases of multiplications is statistically significant ($t = 3.66$, df = 49, $p < .005$). For this reason, the two multiplication cases were studied separately and the results are presented below, separately for each case. Moreover, the time that students needed to find the result of multiplications in case of two-digit multiplicand and multiplier ($M = 20.90\text{sec}, SD = 8.49\text{sec}$) is significantly higher ($t = ...
5.538, df = 49, p < .001) compared with that of a single-digit multiplicand and multiplier (M = 12.94sec, SD = 8.18sec). The success rates both in the case of a multiplication by a single-digit multiplier and two-digit multiplicand (t = 5.597, df = 49, p < .001) and in the case of a multiplication by a two-digit multiplier and multiplicand (t = 11.060, df = 49, p < .001) were significantly lower than the students’ success rates in the multiplication tables. However, there was a statistically significant correlation with success in multiplication tables only in the case of multiplication by a two-digit multiplier and multiplicand (Table 4), which is actually low (r = 0.288, p < .05).

**Table 4.** Correlation success rates in multiplication tables and two-digit multiplications

<table>
<thead>
<tr>
<th>Operations</th>
<th>r</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prep &amp; one two – digit number</td>
<td>.038</td>
<td>.796</td>
</tr>
<tr>
<td>Prep &amp; two – digit numbers</td>
<td>.288</td>
<td>.042</td>
</tr>
</tbody>
</table>

### 3.3.2 Strategies

Concerning the strategies used by the students in order to mentally calculate two-digit multiplications, the following can be highlighted. Table 3 presents the percentages of the main strategies used by the students in the first interview, where they freely chose their preferred strategy to calculate (situation choice). The basic strategy that students use is the mental representation of the written algorithm (e.g., in the operation 9x21), they imagine the vertically written algorithm and calculate: 1x9 = 9, 2x9 = 180, vertically add 9 +180 and they find the result 189. A small number of students use a holistic strategy or compensation strategy, e.g., in the operation 9x21 they convert 9 to 10 and calculate 10x21 = 210, 210-21 = 189 or in the operation 8x99 they count 8x100 = 800, 800-8 = 792. In the second, ‘nochoice’ 1 interview, where students had to calculate using a holistic strategy, the rates of using this strategy are low, with the highest being 50% in the operation 19x30, as shown in Table 3. It is therefore proved that this strategy was not known to the students and after being informed about it, they had difficulty to employ it. Some students also used the strategy of portioning a number, where separate two-digit numbers mainly in units and tens (e.g., in the operation 9x21), they partition 21 into 20 and 1 and multiply 9x21 = 9x (20 +1) = 180 +9 = 189.

### 3.4. Flexibility in mental multiplication of two digit numbers

The percentages of flexible students in relation to each operation are presented in Table 5.

**Table 5.** Frequencies and percentages of flexible students per operation

<table>
<thead>
<tr>
<th>Operation</th>
<th>8x25</th>
<th>9x21</th>
<th>8x99</th>
<th>19x30</th>
<th>11x18</th>
<th>15x49</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effect of flexibility (%)</td>
<td>10</td>
<td>8</td>
<td>4</td>
<td>17</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>(20%)</td>
<td>(16%)</td>
<td>(8%)</td>
<td>(34%)</td>
<td>(12%)</td>
<td>(12%)</td>
</tr>
</tbody>
</table>
It is observed that the percentage of students who are flexible in mentally multiplying two-digit numbers is very small (17%). Only in the operation 19x30 flexible students reached one third of the students (34%), while in the rest of the operations the percentage of flexible students is much smaller, with the smallest percentage being 8% for the operation 8x99. Based on students’ flexibility in mental multiplications, their flexibility in all operations was estimated and was categorized on three levels of flexibility: zero, low and medium. Students classified as of zero flexibility were not flexible in any operation of all given and represent 44% of the sample. Students classified as of low level of flexibility were flexible in only one of the six operations and represent 28% of the sample, while students classified as of medium level of flexibility were flexible in 2.6 out of the six operations and represent 28% of the sample. It is worth mentioning that there were no students to be flexible in all operations, while there was only one student who was characterized as flexible in five out of six operations.

3.4.1 Flexibility and multiplications by single-digit multiplier and two-digit multiplicand

Students’ level of flexibility affects significantly their performance in multiplications by a single-digit multiplier and a two-digit multiplicand ($F_{2, 47} = 4.895, p < .005$). According to the LSD test, students with medium level of flexibility show higher success rates ($M = 85.71\%$, $SD = 17.11\%$) compared to students of low flexibility level ($M = 60.61\%$, $SD = 30.23\%$) or zero flexibility level ($M = 54.76\%$, $SD = 33.61\%$). Moreover, the time that students need to find the result of multiplications by a single-digit multiplier and two-digit multiplicand is significantly influenced by the students’ flexibility level ($F_{2, 47} = 4.246, p < .05$). The analysis continued with the LSD test, and it was indicated that students of medium level of flexibility needed significantly less time to find the result of these multiplications ($M = 8.19$ sec, $SD = 5.84$ sec), compared to students of low flexibility level ($M = 13.73$ sec, $SD = 7.77$ sec) or zero flexibility ($M = 16.45$ sec, $SD = 9.07$ sec).

3.4.2 Flexibility and multiplications by a two-digit multiplier and a two-digit multiplicand

Students’ flexibility level does not affect their performance in case of multiplications by a two-digit multiplier and a two-digit multiplicand ($F_{2, 47} = 0.466, p = .630$). Thus, students, regardless of their flexibility level, show a comparatively low performance in the case of a two-digit multiplier and multiplicand, with an average success rate of 46% and a standard deviation of 30%. Also, the time students need to find the result of these multiplications, does not seem to be significantly affected by the students’ flexibility level ($F_{2, 47} = 3.117, p = .054$), although students of medium flexibility level need less time ($M = 16.29$ sec, $SD = 6.39$ sec) to find the result, compared to students of low flexibility level ($M = 22.69$ sec, $SD = 5.87$ sec) or zero flexibility level ($M = 22.69$ sec, $SD = 10.10$ sec).
4. DISCUSSION AND CONCLUDING REMARKS

To sum up, the results of this research concerning the knowledge of prospective teachers in multiplication tables and mental calculations with two-digit multiplications are the following: the participants were found to ignore the products of some multiplication tables. The most difficult product being 6x9 could not be calculated by 22% and could not be immediately recalled from memory by 36% of students. Also 22% of students could not calculate two out of the 12 given products. Regarding the strategies used, it was observed that the dominant strategy was that of direct recall from memory which was used by 64% to 98% of the students. Other strategies used alternatively by students at much lower rates are the strategy of derived- fact, which was used by 8% of the students and the strategy of recite all multiplication column, which reaches the percentage of 10% for the difficult product 6x9. The products of the multiplication tables with small numbers were revealed to show higher success rate and shorter response time. These results are in accordance with those of by Le Fevre’s et al. study (1996) conducted with introductory psychology students. They reported direct retrieval on approximately 80% of trials, but also reported rules (e.g., anything times 0 is 0), repeated addition (e.g., 2x4 = 4 + 4), number series (e.g., 3x5 = 5, 10, 15), and derived facts (e.g., 6x7 = [6x6] +6).

Concerning the performance and strategies used by students for mental two-digit multiplications we came to the following conclusions. The mental single–digit by two-digit number multiplications presented a significantly higher rate of success than the two-digit by two-digit number multiplications (66% - 44%). Also, single- digit by a two-digit number multiplications present statistically smaller response time by two-digit by two-digit number multiplications. Comparing the results of this study with those of Ligouras’s study (2012) conducted to sixth grade primary school students, who solved almost the same operations, we come to the conclusion that the university students’ success rate in two-digit multiplications (56%) is higher than that of the primary school students (30.49 %).

Regarding the strategies used by the students for the calculation of two-digit multiplications in the free situation choice (choice) the dominant strategy was the mental representation of the written algorithm (48.3%). Meanwhile, a certain number of the students also used the strategy of “partitioning number” (34%) and a small percentage (18%) used holistic strategies. The fact that they did not know how to use holistic strategies was also demonstrated by the success rate (36%) in no choice 1 situation, where they had to use holistic strategies. This percentage is, however, larger than that (15.96%) of the primary school students in the Ligouras’s study (2012). The ignorance of holistic strategies is possibly one of the main causes of low flexibility, with an average rate of 17%, that students present regarding two-digit multiplications use of strategies. This flexibility percentage is greater than the very low flexibility rate (5.92%) of the sixth grade school students in similar multiplications in the research conducted by Ligouras (2012).
Concerning the relationship of success in two-digit multiplications and the success in the multiplication tables we observed that the correlation of prep with a two-digit multiplied by a single-digit number is very small and not statistically significant ($r = 0.038$, $p > .05$), while the correlation of the prep with multiplications of a two-digit by a two-digit number is statistically significant but there is a low correlation index $r$ ($r = 0.288$, $p < .05$). We can therefore assume that for the success in mental two-digit multiplications the knowledge of the multiplication table is not enough. Moreover, students which are efficient in multiplication are not necessarily efficient in mental two-digit multiplications.

The findings also showed that students with medium flexibility in mental calculations can achieve statistically higher success rates at prep and statistically shorter response times in prep with large numbers and marginally shorter response times in prep with small numbers compared to students who are not flexible. So, though it seems that there is no strong relationship between their performance in mental two-digit multiplications with their performance in prep, on the contrary university students with medium flexibility in mental two-digit multiplications show highest success rates and shorter response times. This means medium flexibility students in two-digit multiplications are better at prep and recall it faster from memory than students who are not flexible. Finally, it was revealed that students of medium level of flexibility achieve statistically higher rates and shorter response times in mental single-digit multiplications by a two-digit number. Although students of medium flexibility level do not present statistically higher success rates in mental multiplications of a two-digit by a two-digit, their response time of these students is marginally not statistically smaller than that of students of low or none flexibility.

In relation to the prospective teachers’ skills in mental multiplications we can draw the following conclusions. The majority of the participants seemed not to know the prep and not use direct recall as an exclusive strategy. Le Fevre et al. (1996) also drew these conclusions conducting a study with psychology students.

The majority of the participants did not show a high level flexibility in the strategy use. Also, very low flexibility levels regarding Greek students of the sixth primary school grade were found in Ligouras’s study (2012), for mental multiplications additions and subtractions. Although for the majority of the participants there is no strong correlation between success in prep and two-digit multiplications, students of medium flexibility level achieve statistically higher success rates and shorter response times. In addition, the students of medium flexibility level seemed to tend to have better success rates and response times than students who are not flexible.

**Educational considerations**

Based on the results of this study, we can assume that the participants were not taught in the logic of mental calculations as pupils with the aim to develop flexibility in strategy use. The findings of the study indicated that this lack of flexibility in strategy use, which is evident in prospective teachers’ skills, probably was one of the reason of low success rates at mental two-digit multiplications. In order to
compensate this lack of flexibility related to prospective teachers’ skills, there is an immediate need for delivering training courses integrating theory and practice. For this purpose, it is suggested to think of providing such a training program.

REFERENCES


**BRIEF BIOGRAPHIES**

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INVESTIGATING PROSPECTIVE ELEMENTARY TEACHERS’ NUMBER SENSE, THROUGH MENTAL COMPUTATION STRATEGIES

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ABSTRACT
The purpose of this study was to explore the number sense of prospective elementary teachers, through a test demanding mental computations and estimation strategies. Eighty seven pre-service teachers from the Department of Primary Education participated in this study. Findings were analysed with quantitative methods. At the end of the research, prospective teachers’ number sense was found to be very low, not only in what concerns rational numbers, but even about the basics of decimal system and elementary numerical properties.

Keywords: Number Sense, Mental Computations, Estimation.

1. INTRODUCTION
Prospective teachers, in Greece, enter to the university having completed secondary education, following (mainly) the General Strand (General Lyceum) or the Technical-Vocational one. Following the General Lyceum, they have to choose among 3 pathways: theoretical, practical and technological one. Most of the prospective teachers usually have chosen the “theoretical pathway”, that means emphasis on language and literature and much less emphasis on Mathematics. Especially during the final (third) year of their secondary studies, given that the general lyceum does not function as an independent and self-contained school but has been transformed into a preparatory level for access to higher education, most prospective elementary teachers have “lost any contact” with mathematics, even of the elementary level. So, even if they have completed secondary education, it is not probable that they can deal with elementary mathematics with understanding. This is rather a worldwide phenomenon, given that many studies have shown that the understanding of elementary mathematics subject content of prospective elementary teachers is “rule-bound and thin” (Ball 1990:449). In a research review, Mewborn (2000) showed that the knowledge of primary school teachers have mainly a procedural knowledge of mathematics and lack the conceptual understanding to provide explanations for rules and algorithms. Other researchers have come to the same conclusion (Alajmi & Reys,
2. THEORETICAL FRAME

2.1. What is Number sense?

Number sense is related to a person’s deep understanding of numbers and operations. Though there are numerous studies about students’ ‘number sense’, the term is not yet defined precisely (Hope, 1989:12). It seems that the first allusion to number sense had been made using the term ‘quantitative intuition’ (Carpenter, Coburn, Reys & Wilson, 1976), while the term was clearly described in the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989): “Children with good number sense (1) have well-understood number meanings, (2) have developed multiple relationships among numbers, (3) recognize the relative magnitudes of numbers, (4) know the relative effect of operating on numbers, and (5) develop referents for measures of common objects and situations in their environments” (p. 38). Till then, a number of researchers have described the term, but “no two researchers have defined in precisely the same fashion” (Gersten et al. 2005, p. 296). For Greeno (1991) “number sense is a term that requires theoretical analysis rather than a definition” (p. 170).

Number sense has been described as a mathematical proficiency (Kilpatrick et al., 2001:116) including conceptual understanding of numbers and operations with numbers; procedural fluency to perform operations on these numbers; adaptive reasoning to use different representations and benchmarks to estimate the reasonableness of an answer and strategic competence to apply the knowledge in different contexts and a disposition to make sense of numerical situations. It is a proficiency highly personalized (McIntosh, et al., 1997) with its development being a lifelong process (Reys, Lindquist, Lambdin & Smith, 2007).

There are two main tendencies in the description of the “number sense”:

- a “general description”, and
- a description through the enumeration of the concept’ components.

Examples of the first case are the description of the number sense as a “good intuition about numbers and their relationships. It develops gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms” (Howden 1989, p. 11), or a “deep understanding of number” (Griffin, 2003:306).

McIntosh, Reys and Reys (1992) define number sense as “a person’s general understanding of number and operations, along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful strategies for handling numbers and operations” (p.3) and as “an ability to use numbers and quantitative methods as a means of communicating, processing, and
interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity (makes sense)” (p.4).

Concerning the question of “teachability” of the number sense Van de Walle, Bowman and Watkins (1993) see the developing of number sense as a way of teaching rather, than as a body of knowledge and skills to be taught. This is because they see it as having an interconnected, multi-faceted and highly conceptual nature.

The high conceptual nature of number sense and his link with higher order thinking was referred also by Resnick (1989), who lists the following key features of number sense: number sense is non-algorithmic; tends to be complex; often yields multiple solutions, each with costs and benefits, rather than unique solutions; involves nuanced judgments and interpretation and the application of multiple criteria; often involves uncertainty; involves self-regulation of the thinking process and imposing meaning; number sense is effortful (p. 37).

In his literature review (by analyzing forty studies), Berch (2005) found approximately thirty components of number sense, ranging from the ability to compare quantities, to estimate, to the understanding of number meanings and the effect of operations, to the composing and decomposing numbers, to the skill of having a non-algorithmic “feel” for numbers etc.

McIntosh, Reys and Reys (1992) were the first to provide a framework for clarifying and organising the various components of basic number sense. They proposed three key components to number sense: (i) knowledge of and facility with numbers, (ii) knowledge of and facility with operations and (iii) applying knowledge of and facility with numbers and operations to computational setting, as well as their interconnections (p.5). Each one of these components were further analyzed by other researchers (Reys et al. 1999; Tsao 2005; Yang, Reys, and Reys 2009).

For example, knowledge of and facility with numbers has been analyzed in

1. the understanding of the meaning and (relative) size of number (How does 2/5 compare in size to 1/2?) (Behr, Wachsmuth, Post, & Lesh, 1984; Cramer, Post, & delMas, 2002) and
2. the understanding and use of equivalent representations of numbers/Being able to compose and decompose numbers (Show different ways that 2/5 can be represented )
3. Using measurement benchmarks in comparing numbers

knowledge of and facility with operations, has been analyzed in

1. the understanding the meaning and effect of operations (Is 750: 0.98 more or less than 750?) (Greer, 1987; Graber & Tirosh, 1990; Tirosh, 2000)
2. the understanding and use of equivalent expressions

applying knowledge of and facility with numbers and operations, has been analyzed in
1. Flexible computing and counting strategies for mental computation, written computation, and calculators/apply estimation strategies (Sowder, 1992).

2. Judging the reasonableness of computational results

2.2. Number sense and mental computations

Number sense and mental computation are strongly interrelated: In order for students to use “mental computation strategies flexibly requires sound number sense” and when “students have opportunities to work with numbers in flexible ways, provide opportunities for them to improve their number sense. Needing number sense for efficient use of computation strategies and the development of number sense by using such strategies are very closely interrelated” (Hartnett, 2007:345).

For McIntosh and Dole (2000:407), “it appears that mental computation and number sense need to become integral components of curriculum and assessment procedures, at class, school and system levels. Otherwise, the curriculum may be distorted by playing down the importance of number sense and mental computation, and students may be either advantaged or disadvantaged if there is failure to assess important aspects of mathematics”. By many researchers, mental computation is considered as a subset of number sense, given that when students are proficient in mental computation, they also display number sense (e.g., McIntosh, 1996; McIntosh, Reys, & Reys, 1992; Sowder, 1990). Callingham (2005:193) describes research in mental computation as focusing on “identifying and describing students’ strategies for addressing particular kinds of calculations, often within a framework of number sense” (p. 193). Furthermore, specific connections have been proposed among mental computation and aspects of number sense, in particular, number facts knowledge and estimation (Sowder, 1992). Based on this assumption literature has proposed the importance of including mental computation in a mathematics curriculum that promotes number sense (e.g., Klein & Beishuizen, 1994; McIntosh, 1998; Sowder, 1992; Verschaffel & De Corte, 1996). “Curriculum should provide opportunities for students to develop and use techniques for mental arithmetic and estimation as a means of promoting deeper number sense” (Kilpatrick, Swafford and Findell 2001:415).

Nevertheless, other researchers have argued that proficiency in mental computations helps in developing number sense only if students are encouraged to formulate their own mental computation strategies (Blöte, Klein, & Beishuizen, 2000; Sowder, 1990). “Mental computation can facilitate number sense when students are encouraged to be flexible” (Heirdsfield, 2004:443). Even in this case, the high scores in mental computations is not an indice of the number sense. It is the kind of strategies used by the students that guarantees their sense of number: “Students who may score highly on mental computation tests and general mathematics tests may not be developing a "sense" of numbers. And students who do not score highly on written tests of mental computation, number sense and general mathematics may
still have quite good strategies for mental computation and a lot of "sense" about numbers” (McIntosh and Dole, 2004:407).

2.3. Number sense of prospective elementary teachers

Although relatively few studies have investigated pre-service teachers’ knowledge of and facility with numbers and operations comparing those conducted in order to investigate students’ number sense, the findings indicate the low performance of teachers. Even if prospective elementary teachers were able to manipulate symbols algorithmically and find mathematical products, they were unable to create intuitive algorithms, arguments, or models that rely on number sense and mathematical reasoning. Kaminski (1997) in a small-scale study (with 6 pre-service teachers) on the use of number sense in the whole number domain, found that teachers preferred using written calculations and rarely utilized estimation, they lacked an understanding of multiple relationships in the number and operations domain and had difficulties with mental computations. Johnson (1998) arrived in the same conclusion: prospective elementary teachers' general number sense are inadequately developed, they resist looking at mathematics in creative, non-algorithmic ways. Tsao (2004) explored the connections between number sense, mental computations and written computations of 155 pre-service elementary school teachers and found that the correct responses on exact computations were higher than those requiring mental computation, estimation or other aspects of number sense. The same researcher (Tsao 2005), in a qualitative research conducted with 12 pre-service elementary school teachers, explored five characteristics of number sense: the ability to decompose/ recompose numbers; recognizing the relative and absolute magnitude of numbers; the use of benchmarks; understanding the relative effect of operations on numbers; and flexibility of applying the knowledge of numbers and operations to computational situations (including mental computation and computational estimation). He found that prospective elementary teachers exhibit poor number sense and rely heavily on standard written algorithms. A most focus studies were organized by Alajmi and Reys (2007) and Yang, Reys and Reys (2009). Alajmi and Reys investigated 13 middle school teachers’ capacity to determine the reasonableness of answers. They found that the common view of the teachers of a reasonable answer was an exact answer and that they would use a computational procedure to determine the reasonableness of an answer. The population in the study of Yang, Reys and Reys (2009) was much greater. 280 pre-service elementary teachers were tested about their competency of using benchmarks in recognizing the magnitude of numbers and estimation in knowing the relative effects of an operation on various numbers. Over 60% of the teachers failed to use attributes of number sense - benchmarks and estimation - to produce answers and explain their thinking. Prospective elementary teachers have tended to rely on standard written algorithms, despite the fact that the instructions for these assessments explicitly discouraged such approaches. Şengül, S. (2013) has investigated five different number sense
components. His population was 133 prospective teachers from the Elementary Education Department and the findings were analyzed with qualitative and quantitative methods. Analyzing the results he concluded that pre-service teachers’ number sense was very low, and that pre-service teachers preferred using “rule based methods” instead of “number sense” in each of the components”. In a recent study, Tsao (2012) investigated the number sense of teachers with different backgrounds: mathematics and physics backgrounds, and language backgrounds. He observed that “the teachers with mathematics and physics backgrounds had more complete interpretation towards the related knowledge of number sense. The teachers who did not have mathematics and physics backgrounds but were extremely interested in mathematics had initial understanding towards the related knowledge of number sense. The teachers without mathematics and physics were not interested in mathematics and had less knowledge of number sense” (p.29). A special interest for our study presents the research of Lemonidis and Kaimakami (2013) on prospective elementary teachers’ knowledge in computational estimation. Their main result was that “Greek prospective elementary teachers show a low performance in number sense. [...] show a general limited ability in performing mental operations, especially in the case of two-digit number multiplications and in the case of divisions with two-digit divisor “(p.96).

3. METHOD

3.1. Aims of the study

The present study were designed in order to (1) investigate prospective elementary teachers primary teachers number sense as it is defined by (McIntosh, Reys and Reys 1992) and (2) detect the major errors/misconceptions concerning the meaning of numbers and operations on numbers.

3.2. Sample

Participants were 87 prospective teachers in the first year of their studies in University of Patras -Department of Primary Education. The sample consisted of 9 male (10.3%) and 78 female (89.7%) students. The 74 students (85%) were attending in General Lyceum a theoretical pathway, the 6 of them (6.8%) the practical pathway and the rest 7 of them were students having accomplished a circle of studies in another department (8%).

3.3. Instrument development

The design of test items was based on the conception of number sense as an understanding of (McIntosh, Reys and Reys 1992):

a. the meaning and the relative size of numbers
b. the meaning and the relative effect of operations on numbers.
c. the different ways of applying knowledge of and facility with numbers and operations

The 28 items were constructed in order to investigate prospective elementary teachers’ number sense, being analyzed in the 4 number sense components as presented in the following table. Prospective teachers had 50’ to answer the test and the only indication given was to «answer with written calculation, only mentally computing». The time restriction was decided to prevent the use of written computations.

<table>
<thead>
<tr>
<th>Number sense components</th>
<th>Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1. Understanding of the meaning and (relative) size of number-Sense of orderliness, place value, order numbers</td>
<td>Questions 1, 2, 4, 10, 13, 19, 20, 25</td>
</tr>
<tr>
<td>Using measurement benchmarks in comparing numbers</td>
<td></td>
</tr>
<tr>
<td>A2. Understanding and use of equivalent representations of numbers (equivalent numerical forms) /Being able to compose and decompose numbers</td>
<td>Questions 7, 8, 9</td>
</tr>
<tr>
<td>B. The meaning and the relative effect of operations on numbers.</td>
<td>Questions 3, 5, 6, 16, 17, 21, 22, 26, 27</td>
</tr>
<tr>
<td>C. The different ways of applying knowledge of and facility with numbers and operations.</td>
<td>Questions 11, 12, 15, 18, 23, 24, 28</td>
</tr>
</tbody>
</table>

**4. RESULTS**

**4.1. Data**

The data was analyzed by using SPSS and Microsoft Excel. Every answer was evaluated as right, wrong or no answer. Furthermore, the answers were encoded in “1” for the right ones and in “0” for wrong or no answers. The mean of every answer was computed, by dividing the score in every answer with the whole sample. The results (in frequency and percentage of the right answers) of the ‘number sense’ test are displayed in Table 1 for the total number sense and for the four selected domains.

<table>
<thead>
<tr>
<th>Table 1: Percentages and frequencies of success</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>DOMAIN 1: MEANING AND SIZE OF NUMBER</strong></td>
</tr>
<tr>
<td>Q. 1</td>
</tr>
<tr>
<td>Success</td>
</tr>
<tr>
<td>(24.1%)</td>
</tr>
<tr>
<td><strong>Domain 2: UNDERSTANDING AND USE OF EQUIVALENT REPRESENTATIONS OF NUMBERS</strong></td>
</tr>
<tr>
<td>Q. 7</td>
</tr>
<tr>
<td>Success</td>
</tr>
<tr>
<td><strong>DOMAIN 3: MEANING AND RELATIVE EFFECT OF OPERATIONS ON NUMBERS</strong></td>
</tr>
<tr>
<td>Q. 3</td>
</tr>
<tr>
<td>Success</td>
</tr>
</tbody>
</table>
Eugenia Koleza, Maria Koleli
INVESTIGATING PROSPECTIVE ELEMENTARY TEACHERS’ NUMBER SENSE, THROUGH MENTAL COMPUTATION STRATEGIES

(95.4%) (52.9) (63.2% (51.7% (55.2%) (48.3% (37.9%) (79.3% (60.9%)

DOMAIN 4: DIFFERENT WAYS OF APPLYING KNOWLEDGE OF AND FACILITY WITH NUMBERS AND OPERATIONS

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Success</td>
<td>30</td>
<td>33</td>
<td>77</td>
<td>53</td>
<td>36</td>
<td>21</td>
<td>30</td>
</tr>
<tr>
<td>(34.5%)</td>
<td>(37.9%)</td>
<td>(88.5%)</td>
<td>(60.9%)</td>
<td>(41.4%)</td>
<td>(24.1%)</td>
<td>(34.5%)</td>
<td>(65.5%)</td>
</tr>
</tbody>
</table>

The mean score of each domain was computed by summing the frequencies of each domain and dividing with the sample (87) and the number of the answers in each domain. After that, the mean of overall score was computed too, by summing all the frequencies and dividing them with the number of all questions (28) and the sample (87). The means’ distributions appear in Figure 1.

**Figure 1: Means of Questions**

Furthermore, the overall score and the four domains were categorized in 4 values due to the number of the right answers. The selected values are the ‘none’, ‘poor’, ‘average’ and ‘good’.

- None: no right answers
- Poor: right answers in the 1/3 of the questions
- Average: right answers in the 2/3 of questions
- Good: right answers in more than the 2/3 of questions.

The overall score cannot take the value “none” as every member of the sample answered right in more than 4 questions. On the other hand, there were members of the sample that didn’t answer correctly any of the questions in one particular domain,
so the value “none” appears in all the four domains. This type of analysis appears in Figure 2 and Table 2.

**Figure 2:** The relation between the overall score and the four number sense components

For example, in the category “Meaning And Relative Effect Of Operations On Numbers”: One student (1.1%) gave no right answers in all the questions of this category and so, her overall score was poor. Seventeen students (19.5%) answered correctly to 1/3 of the questions (‘poor’), and among them fourteen (16.1%) had a ‘poor overall score’ and three (3.4%) had an ‘average overall score’. Thirty eight (43.7%) students had an average performance in this domain, as about their overall performance, six (6.9%) had a poor one, twenty nine (33.3%) had an average one and three (3.4%) had a good overall performance. Finally, thirty one (35.6%) students answered good in the questions of this category. Among them, eighteen (20.7%) had an average and thirteen (14.9%) had a good overall performance.

**Table 2:** The relation between the overall score and the four number sense components

<table>
<thead>
<tr>
<th>MEANING AND SIZE OF NUMBER</th>
<th>NONE</th>
<th>POOR</th>
<th>AVERAGE</th>
<th>GOOD</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>OVERALL SCORE</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>POOR</td>
<td>3</td>
<td>18</td>
<td>0</td>
<td>0</td>
<td>21</td>
</tr>
<tr>
<td>AVERAGE</td>
<td>0</td>
<td>43</td>
<td>7</td>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td>GOOD</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>8</td>
<td>16</td>
</tr>
</tbody>
</table>
The above results allow us to categorize all questions of the test in 4 groups depending on the success in each.

- **Group A** (more than 75%): 3, 14, 26
- **Group B** (50%-75%): 2, 4, 5, 6, 9, 15, 16, 17, 27, 28
- **Group C** (25%-50%): 7, 8, 10, 11, 12, 13, 18, 19, 20, 21, 22, 24, 25
- **Group D** (less than 25%): 1, 23

---

**Percentages per question**

![Bar chart showing percentages per question]
Prospective teachers’ percentage of correct responses range from 24.1% to 73.6% in Domain 1 (Meaning and size of number) 35.6% to 67.8% in Domain 2 (Understanding and use of equivalent representations) 37.9% to 95.4% in Domain 3 (Meaning and relative effect of operations) 24.1% to 88.5% in Domain 4 (Applying knowledge and facility with numbers)

The overall percentage of correct responses for all ‘number sense’ domains is 50%, actually not that high.

Analyzing teachers’ responses we made two key observations:
1. Half of the items in the questionnaire (15/28) had been answered correctly by less than the 50% of the prospective teachers
2. The mathematical content of these items were:
   - Positional system: Q1
   - Properties of mathematical operations: Q7, Q8
   - Multiplication/Division between decimals: Q18, Q23, Q24
   - Rational numbers: Q10, Q11, Q12, Q13, Q19, Q20, Q21, Q22, Q25

None of the questions were answered by more than 50% of the participants. The rest 13 questions were about integers and only two of them were about a division of an integer by a decimal.

Though it appears that prospective teachers performed the best in the “meaning and relative effect of operations” domain (means 61%), in fact, this high mean is due to the questions 3 and 26.

Question 3 (95.4%): “If you know that 48+ 37=85, find without using any calculation how much is 49+36?” is a relatively easy ‘algebraic’ question that students deal with during the second grade.

Question 26 (79.3%),
“Choose the greater of the two:
   a) 135+98 or 114+92
   b) \( \frac{1}{2} - \frac{3}{4} \) or \( \frac{11}{2} \)
   c) 46-19 or 46-17
   d) 0,0358 or 0,0016+0,313”

also, deal with simple mental calculations, given that in (a) they could replace 98 and 92 and make mentally the addition, in (b) the comparison of 3/4 and 1 is elementary, in (c) they had to notice that we take a greater result when we subtract a smaller number and in (d) they should notice that in the addition there is the number 0.313 that is greater than 0.0358.

From the other Domains, the questions with the higher success that influenced the means of the domain were:
- For Domain 1 (Meaning and size of numbers),

The Question 2 (73.6%): “How many notes of 100€ you can have in 5908 €?”. It is a real life situation very familiar to the adult population.
All other questions of the domain had a percentage success below 50%.

- For Domain 2 (Understanding and use of equivalent representations),
  The Question 9 (67.9%):

  ![Number line diagram](image)

  On this number line, which letter represents better the calculation B+ F? Explain why.

  B is approximately 0.2 and F is approximately 2.2. The sum of B+F is lower than 3 so the answer is the letter G. Some of the students answered both G and H because the result should be greater than F.

- For Domain 4 (Different ways of applying knowledge...),
  The numbers in the Question 14 (88.5%): “The product of which two of the following numbers is closer to 75? 4, 18, 50, 37” lead easily to the correct answer.

  The same thing happens to the Question 28 (65.5%): “Find mentally the products a)18×5×5×2×2 b)25×62×4 c)2,5×0×4.

  In (a) the 2×5=10 and in (b) the 4×25=100 has facilitated the answer.

  In the Question 15 (60.9%): “The product of 46×91
  - is greater, equal or lower than 5000?
  - is greater, equal or lower than 3600? “

  In the first part of this task, they had to notice that 46×100= 4600 so the result would be lower than 5000. In the second part of this task, they had to notice that 40×90= 3600 so the result would be greater than 3600.

  All other questions of this domain had a percentage success below 38%.

  We give some examples of the most common mistakes made by the prospective teachers in these 15 low achieving questions. We remind that in all questions we asked for the explanation of their way of thinking.

<table>
<thead>
<tr>
<th>QUESTIONS</th>
<th>COMMON ERRORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) How many tens and hundreds has the number 1457?</td>
<td>5 tens and 4 hundreds</td>
</tr>
<tr>
<td>7) Which of the following calculations has the same result as the product 4×198;</td>
<td>a) 4×200-2</td>
</tr>
<tr>
<td>a) 4× 200 – 2 b) 4× 200 – 4 c)4× 200 – 6 d) 4× 200 – 8</td>
<td></td>
</tr>
<tr>
<td>8) We know that 93× 134 =12462. How much more than 12462 is the product of 93× 135?</td>
<td>12465 or 3 more because when we multiple 3 with 5, the product has to end in 5.</td>
</tr>
<tr>
<td>18) Our calculator is broken and it gives 6858 as the result of the product 15.24 × 4.5. Where should we place the decimal point in order to have the correct result?</td>
<td>6.858 because we have 3 decimal digits</td>
</tr>
</tbody>
</table>
QUESTIONS

23) The product of 0.048 \times 0.19 is closer to:  
a) 0.00009, 
b) 0.0009, c) 0.009, d) 0.09  
a) 0.00009 Because the sum of the decimal digits are five.

24) The result of 4.2 : 0.33 is closer to:  
1) 0.012, b) 0.12, c) 1.2, d) 12  
a) 0.012 and b) 0.120 because we want 3 decimal digits

10) Which of the following fractions is closer to  
1) 4/5, 2) 6/7, 3) 8/9  
4/5 because the whole is divided in bigger parts

11) Which of the following numbers is closer to  
1) 7/12 + 3/8; 2) 1 1/2; 3) 0.1  
a) 1 b) 2 c) 19 d) 21  
a) 1 because 12/13 and 7/8 are close to number 1 so the sum would be close to 1.

12) Which of the following results is closer to 1?  
b) 7/15 + 5/12 because these fractions have the biggest denominators.

13) Use ONLY 2 of the numbers 3, 4, 9, 12, to create a ratio with value closer to  
1) 1/2  
3/4 because it is really close to 2/4=1/2

19) Compare mentally the fractions below  
1) 13/28 vs 15/32, 2) 8/9 vs 4/5, 3) 16/19 vs 15/18  
3/4 because it is really close to 2/4=1/2 and really close to 1/2

20) Which of the fractions below is the greatest?  
4) 4568/4569, B. 4569/4570, C. 499/500, D. 500/501  
c) 499/500 because in all fractions the numerator is by 1 less from the denominator, so the fraction with the lower denominator (bigger pieces) is chosen.

21) Without computing the product 1/2 \times 3/4 can you predict if the result is:  
a) greater than 1/2, b) lower than 1/2, c) greater than 1, d) greater than 3  
c) greater than 1 because of the multiplication of fractions that are greater and equal of 1/2

22) Without actually computing 2/3 : 1/3 can you predict if the result is:  
a) greater than 1, b) lower than 1, c) lower than 1/3, d) greater than 3  
b) lower than 1 because of the division of fractions.

25) The sum of 12/14 + 7/8 is closer to:  
a) 1, b) 2, c) 19/21, d) 91/100  
c) 19/21 because 12/13+7/9=19/22

5. DISCUSSION
Have primary prospective teachers a ‘number sense’?

The results of our study lead us to give a rather negative answer to this question. The non-sense of numbers among the prospective teachers becomes particularly obvious by their inability to distinguish between the place-name and the place-value of digits in a number (see Question 1). An explanation of their misconception is probably the one given by Sowder (1997:449), that “place-value instruction is traditionally limited to the placement of digits. Thus, children are taught that the 7 in 7200 is in the thousands place, the 2 is in the hundreds place, a 0 is in the tens place, and a 0 is in the ones place. [...] They do not read the numbers as 7200 ones or 720 tens, or hundreds, and certainly not as 7.2 thousands”. The remedial proposed by Sowder in the same paper is -before we begin instruction on decimal numbers-, to provide more instruction on place value with whole numbers, and to practice reading numbers in different ways.

The results have also made apparent prospective teachers’ ignorance of the basic numerical operations. About 55% of teachers consider that $4 \times 198= 4 \times 200 - 2$ (Question 7), and 65% of teachers are not able to read $93 \times 134 = 12462$ as 134 times 93 t, so as to conclude that $93 \times 135$ is 93 more (Question 8).

Results have also made obvious prospective teachers’ non sense of decimals and fractions. For example, (Question 24) they are unable to estimate that 0,33 is approximately 1/3, so the result of 4,2: 0,33 is closer to 1,2. For their answer they take into consideration only the sum of the decimal digits as if it was about an operation of addition or substraction.

Furthermore, in their majority, they have no sense of fractions, considering as a condition of equality the “same difference” between the nominator and the denominator. As about the way of adding fractions, the younger students’ common mistake of adding nominators and denominators appear to the prospective teachers also. In fractions’ multiplication they apply the “tacit rule” that “multilication makes bigger if the multiplier is bigger than the multiplicand”. In the Question 21 they consider the product $\frac{1}{2} \times \frac{3}{4}$ as bigger than 1/2 because 3/4 is bigger than the 1/2.

How can teachers help students acquire a “number sense”? Creating in the classroom an environment that fosters curiosity and exploration at all grade levels, and taking into consideration that number sense cannot be a goal of direct instruction (Greeno 1991:173) It "develops gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms" (Howden 1989:11).

But, to be able teachers to help their students acquire a number sense, they must themselves have the “sense” of numbers. The results of this study seems to indicate that this pre- condition is not obvious.

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**Maria Koleli** is an in service teacher. She completed her studies in the Pedagogical Faculty of Primary Education in University of Patras. Her scientific interests are: Mathematics in everyday life, mental calculation and studying the use of technology in teaching/learning of mathematics.
ELEMENTARY TEACHERS’ EFFICIENCY IN COMPUTATIONAL ESTIMATION PROBLEMS

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ABSTRACT
The aim of this study was to explore in-service teachers’ performance and strategies in computational estimation and possible individual difference. Additionally, the study looked for possible individual difference in terms of age and work experience, their attitude towards mathematics and their prior performance in mathematics during high school years. Eighty Greek in-service teachers participated in the study. Results showed that in-service teachers performed quite well, in most of the computational estimations and used a variety of strategies. Individual differences were captured only in terms of in-service attitude towards mathematics.

Keywords: Computational estimation, in-service teachers, individual differences strategies in computational estimation.

1. INTRODUCTION
Improving the teaching of computational estimation is considered as the key to encourage students’ sense of number development in students (Edwards, 1984; Greeno 1991; McIntosh 2004; Tsao, 2009). NCTM (2000) deemed computational estimation as an important skill for students to become proficient in mathematics. There is wide agreement among mathematics’ educators that the ability to judge the appropriateness of results of computations by computational estimation may be more important and practical than the exact calculation for many everyday mathematical situations. In mathematics curriculum standards experts worldwide stressed the importance of computational estimation instruction (Australia: Australian Education Council, 1991; USA: National Council of Teachers of Mathematics, 2000; Taiwan: Ministry of Education, 2003). Computational estimations were introduced in Greece with the reform of 2006. However, since then, a systematic
teaching or training for teachers lacks and so, teachers still use in their everyday practice traditional teaching, giving greater emphasis to standard algorithms with pencil and paper than to mental calculations that are at the core of computational estimation. This could be due to the fact that teachers lack the knowledge to perform computational estimations and to use a number of strategies that are appropriate for executing computational estimations. Indeed, previous studies have already showed pre-service teachers’ lack of knowledge in computational estimations (Lemonidis & Kaimakami, 2013). However, pre- and in-service teachers were found to have some crucial differences in their performance on computational estimations while this difference depends on the mathematic action that is examined (Tsao, 2013). This study aimed to explore the in-service teachers’ efficiency in performing computational estimations and the strategies they use for this purpose. It is expected that the study of in-service teachers’ computational estimations could provide evidence for understanding why teachers do not implement the computational estimations in their everyday practice and for developing in-service training programs.

1.1. Strategies in computational estimation

Reys, et al., (1982), in their strategies’ analysis of computational estimation, identified three high-level cognitive processes that are intertwined with these strategies: reformulation (the individual modifies the numerical data in order to create a form that is more manageable, leaving the structure of the problem intact), translation (the individual modifies the structure of the problem to generate a more manageable form of the problem) and compensation (the individual proceeds to adjustments on the data in order to reflect the numerical variation that came out from the translation or the reformulation of the problem). Lemonidis (2013), in his review presented a more detailed list of strategies on computational estimation: rounding, front – end strategy, truncating, clustering or averaging, prior compensation, post compensation, compatible numbers strategy, special numbers strategy, substitution, factorization, distributivity and algorithm (see also Dowker, 1992; LeFevre et al., 1993; Reys et al., 1982; Reys et al., 1991; Sowder & Wheeler, 1989) which individuals apply when they solve computational estimation problems with different demands. An important issue is whether the individuals choose to use the appropriate strategy for the particular problem, in the specific context (Verschaffel, Luwel, Torbeyns, & Van Dooren, 2007).

1.2. Factors affecting the estimation ability

The computational estimation is a complex process involving cognitive and emotional components (Liu, & Neber, 2012). Developmental research showed that computational estimation begins surprisingly late and proceeds amazingly slow with gradual improvement after the third and fourth grade (Siegler & Booth, 2005). Sixth graders and adults were more correct than the fourth graders in a sum estimation problem of two three-digit addends (Lemaire & Lecacheur, 2002). Finally, eighth
graders are more efficient in multiplication estimation than the sixth graders, while adults are more correct than the eighth graders (LeFevre et al., 1993).

The efficiency of computational estimation is a constructive procedure with prior knowledge being important in order to develop some more sophisticated strategies, suitable for the particular problem and specific context. Star et al. (2009) showed, in an experimental design research, that students with high computational capacity assessment in pre-test, developed strategies that lead to more precise estimates, while low ability students developed strategies that were easier to perform. Therefore, the prior knowledge has an important role in students’ progress in a higher level of computational estimation (see also Gliner, 1991; Liu & Neber, 2012; Lemaire & Lecacheur, 2002; Lemaire, et al., 2004; Tsao, 2013). This improvement, however, might be a result of the higher cognitive abilities of the individuals that had facilitated the acquisition of the strategies before and during the intervention and helped students benefit more from the intervention.

The efficiency in computational estimation is not an all or nothing phenomenon. Individuals’ efficiency varies between the operations. In multiplication and division, estimation is usually more difficult among the operations, because it is necessary to take into account the effects of operations on the relative size of numbers. For instance, pre-service teachers (Tsao, 2013), as soon as high school students (Bana & Dolma, 2004), and undergraduate students (Hanson & Hogan, 2000) performed significantly better in addition and/or subtraction than in multiplications and divisions. Moreover, the efficiency of individuals in the computational estimations depends on the kind of numbers that are involved in the problem. Fifth graders (Tsao & Pan, 2011) and high school students (Bana & Dolma, 2004) are more efficient in estimation problems with whole numbers than in problems involving decimals and fractions. Additionally, pre-service teachers had more difficulties in estimating problems with fractions than with decimal numbers (Tsao, 2013).

1.3. Surveys to teachers about computational estimation

Although professional mathematicians are very efficient to computational estimations (Alajmi, 2009; Dowker, 1992), both pre-service (Castro, et al., 2002; Gliner, 1991; Goodman, 1991; Lemonidis, & Kaimakami, 2013; Tsao, 2013; Yoshikawa, 1994) and in-service teachers (Alajmi, 2009; Dowker, 1992; Mindenhall, et al., 2009; Tsao & Pan, 2013) efficiency in computational estimations is moderate or low. For instance, Castro, et al. (2002), studied the difficulty of computational estimation tasks— with operations without context— in connection with operation type— multiplication and division— and number type— whole, decimal greater than one and decimal less than one— that involved in them. An estimation test was administered to the teachers and some of them were selected to be interviewed. Castro et al., (2002) concluded that estimating with decimals less than one is more difficult for pre-service teachers than estimating with whole numbers or decimals greater than one. Most errors were produced in estimation processes, due to the teachers’ misconceptions of operations.
of multiplication and division. Additionally, Lemonidis and Kaimakami (2013) studied performance, errors and computational estimation strategies used by 50 pre-service Greek teachers. They found that, pre-service teachers were low level estimators facing more difficulties in multiplication and division estimation problems than in addition problems. Moreover, teachers weren’t familiar and they didn’t use strategies such as averaging and compatible numbers strategy. Tsao and Pan (2013) studied the understanding and the knowledge of practicing teachers in Taipei in computational estimation and the instructional practices which are used by teachers in their everyday teaching practice. Six (three teachers with mathematics/science major and three teachers with non-mathematics/science major) fifth-grade elementary teachers were participating in this study. The findings showed that all teachers were able to explain the meaning of computational estimation, and they efficiently used computational estimation strategies to solve problems. Their computational estimation strategies to solve problems included front-end, rounding, compatible number, special number, use of fractions, nice number and distributive property strategies. All six teachers used special numbers (1, 0, ½), and five of them used rounding and compatible number strategies. Four teachers used nice numbers while only one teacher used front-end strategy and the distributive property. Alajmi (2009) examined 59 elementary and secondary education mathematicians’ strategies on the computational estimations in Kuwait. He found that, albeit some teachers ignored what the computational estimation is, the majority of teachers used rounding in their computational estimations while only 40% of strategies were used effectively, with 76% of those strategies used by secondary teachers.

Scholars also investigated teachers’ attitudes towards computational estimations and how these could affect their everyday practice. Although teachers recognize the usefulness of computational estimation on the daily life, there is not agreement between them for integrating computational estimations in mathematics education. For instance, in the Alajmi’s (2009) study, although two thirds of the teachers considered estimation as an important skill for life, nearly half of them do not consider computational estimation as a significant topic in mathematics education. Tsao (2013) examined 84 pre-service elementary teachers’ attitudes towards computational estimation in Southern Minnesota with Computational Estimation Attitude Survey (CEAS) and their relations with their efficiency in computational estimations. The results showed a relationship between pre-service elementary teachers’ computational estimation and their attitudes towards computational estimation; those who scored higher in computation estimations consider them as necessary, useful, and beneficial for life. Moreover, the relationship with mathematics is correlated with instructional practices (Tsao & Pan, 2013).

1.4. The current study

Although we have an idea of Greek pre-service teachers’ efficiency in performing computational estimations (e.g. Lemonidis & Kaimakami, 2013) we know nothing
about—Greek in-service teachers’ efficiency. This knowledge is important for understanding teachers’ reluctance to implement computational estimations in their classes, and as previous studies have shown, some significant changes between pre- and in-service teachers’ efficiency, on strategies used and on attitudes towards computational estimations have been noticed. Thus, the aim of the current study is to explore the in-service teachers’ efficiency in performing computational estimations and the strategies use. Therefore, this study addressed two main research objectives.

1. To capture Greek in-service teachers’ efficiency and strategies used in performing computational estimation problems.

2. To examine the possible individual differences in terms of personal characteristics of the teacher (i.e. age, sex, teaching experience, prior experience and familiarity with mathematics, and emotional relationship with mathematics) and in terms of the characteristics of particular problems, more specifically the types of operations and the kind of numbers.

2. METHOD

2.1. Participants

Eighty (46 females) in-service Greek teachers participated in the study with average age 40.86 years and average teaching experience 16.51 years. Almost half of them (47.5%) were familiar with mathematics as they attended the high school programs with major mathematics and science and the rest with major humanities. The upper percentile point (25%) of their grades in mathematics at the end of the high school was 19.00, and lower (75%) percentile at the 17.50, and the 50th percentile was 18.00 in the 0 to 20 grading system with 20 to be 100% efficiency. Finally, 58 (72.5%) in-service teachers reported that they are emotionally positive related to mathematics denoting very good or good emotional relationship.

2.2 Sampling method

The sampling method used in this survey was stratified sampling. This method was chosen to ensure the representation of each segment of the population, to reduce the estimation error and to ensure the existence of a sufficient number of subjects from subpopulations. In this method, the population is divided into layers and then selected subsamples with simple random sampling within each stratum.

The layers used, were constructed for the needs of this survey. More specifically, an equal number of men and women teachers was selected (layer by gender), also an equal number of teachers who attended in the high school programs with major mathematics and teachers who attended in high school programs with major humanities (layer based on the major subject attended in the high school). Even age was another criterion for creating a layer, so equal number of young and older educational age was selected.
2.3 Procedure

Participants were executed in 10 computational estimation problems with paper and pencil procedure in the presence of the researcher. They were asked to write on the paper their on-line thoughts during their effort to solve the problem in order these scripts to be informative for the strategies participants used to solve the problems. Additionally, ten of them were examined by interview.

2.4 Tasks

The 10 computational estimation problems addressed to our participants were designed to be solved by using one of the five appropriate strategies, namely front – end strategy, clustering or averaging, rounding with post compensation, compatible numbers strategy and special numbers strategy. The following ten computational estimation problems were set to in-service teachers:

P1. Give an approximate estimate of the sum of the following amounts of money: 1.26 €, 4.79 €, 0.99 €, 1.37 €, 2.58 €

P2. Give an approximate number of students attended in all three schools:
   A secondary school: 1,378 students, B secondary school: 236 students, C secondary school: 442 students

P3. Six student groups prepared flower bouquets for the school feast. The groups prepared 27, 49, 38, 65, 56, 81 flower bouquets. How many flower bouquets have been prepared approximately?

P4. A train of modern technology runs, 25,889 kilometers in 52 hours. How many kilometers does the train cover approximately in one hour?

P5. Is the following result, approximately 200? 35 + 42 + 40 +38 +44.

P6. Six independent measurements were made by the team in order to find the height of mountain Everest: 28,990ft, 28,991ft, 28,994ft, 28,998ft, 29,001ft, 29,026 ft.
   Based on these measurements, what is the approximate height of mountain Everest?

P7. In 816 ml of a substance 9.84% is alcohol. How much alcohol is approximately in the substance?

P8. Mary ran ½ km in the morning and 3/8 Km in the afternoon. Did she run at least 1 km?

P9. A worker worked 28 days for 56 € a day. How much will he approximately be paid?

P10. A student who started ski lessons, completed 75 hours and the cost for each hour was 36€. How much does he have to pay?

Appropriate strategies to solve problems:

- The most appropriate computational estimation strategy for problems P1 and P2 is the front - end strategy. For example, at the problem P1, to calculate the sum of amounts 1.26 €, 4.79 €, 0.99 €, 1.37 €, 2.58 €, is initially calculated the
front end: \(1 + 4 + 1 + 2 = 8\) € and then, back parts of numbers \(0.26 + 0.79 + 0.99 + 0.37 + 0.58 \approx 3\) €, so \(8 + 3 = 11\) €.

- The most appropriate computational estimation strategy for problems P3 and P4 is compatible numbers strategy. For example, at the problem P3, to calculate the sum of 27, 49, 38, 65, 56, 81 we group compatible numbers to create hundreds, 27 + 81 = 100, 38 + 65 = 100, 49 + 56 = 100, so the total will be 300.

- The most appropriate computational estimation strategy for problems P5 and P6 is clustering or averaging strategy. For example, to calculate the sum 35 + 42 + 40 + 38 + 44 at the problem P5, we consider that all the numbers are 40 and counting 40\(\times5 = 200\).

- The most appropriate computational estimation strategy for problems P7 and P8 is special numbers strategy. At the problem P7, for example, to calculate the 9.84% of 816 ml, one can calculate the 10% of 816 ml which is easier. The special number is the 10%.

- The most appropriate computational estimation strategy for problems P9 and P10 is rounding with post compensation strategy. At the problem P9, to calculate multiplying 28\(\times56\), is rounded up 30\(\times60 = 1800\) and after compensation downward 30\(\times4 = 120\), 1,800 - 120 = 1,680.

Additionally, participants were asked to denote their age and sex, the years of their work experience, whether they attended courses at the high school with mathematics as major, and finally they were asked to denote their emotional relationship with mathematics (1 = very bad, 5 = very good) in a 5-point Likert type scale.

3. Results

3.1. Teachers’ efficiency and strategies used by teachers

<table>
<thead>
<tr>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
<th>P5</th>
<th>P6</th>
<th>P7</th>
<th>P8</th>
<th>P9</th>
<th>P10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Performance</td>
<td>f</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>f</td>
<td>f</td>
<td>f</td>
<td>f</td>
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<td>%</td>
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<td>%</td>
<td>%</td>
<td>%</td>
<td>%</td>
<td>%</td>
</tr>
<tr>
<td>Success with estimation</td>
<td>93.8</td>
<td>93.8</td>
<td>91.3</td>
<td>86.2</td>
<td>85</td>
<td>77.5</td>
<td>73.8</td>
<td>72.5</td>
<td>67.5</td>
</tr>
<tr>
<td>Exact calculation</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>2</td>
<td>11</td>
<td>2</td>
</tr>
<tr>
<td>with algorithm</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>2.5</td>
<td>1.3</td>
<td>6.3</td>
<td>2.5</td>
<td>13.8</td>
<td>2.5</td>
</tr>
<tr>
<td>Wrong answer</td>
<td>4.9</td>
<td>2.4</td>
<td>4.9</td>
<td>8.8</td>
<td>2.4</td>
<td>16.2</td>
<td>16.2</td>
<td>2.4</td>
<td>28.7</td>
</tr>
<tr>
<td>No answer</td>
<td>2.4</td>
<td>2.4</td>
<td>2.4</td>
<td>11.3</td>
<td>7.5</td>
<td>11.3</td>
<td>1.3</td>
<td>2.4</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>80</td>
<td>80</td>
<td>80</td>
<td>80</td>
<td>80</td>
<td>80</td>
<td>80</td>
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<tr>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
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<td>100</td>
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<td>100</td>
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<td>100</td>
</tr>
</tbody>
</table>
Table 1 presents the frequencies and percentages (%) of teachers’ accurate responses in computational estimation problems. A small percentage of in-service teachers varying form 1.3 to 13.8%, used the algorithm to make the estimations. For example, in problem P8, there were some teachers who chose to change both fractions to the same denominators and add them then. Others preferred to divide 1,000 meters in 8 pieces, from which they took 3, just to find the correlation of 3/8 to 1 km. Three clusters of problems could be defined in terms of the percentage of teachers that solve accurately the problem; problems of high difficulty, problems of medium difficulty and easy solved problems. Teachers were found to be less accurate in problems demanding an operation of multiplication with numbers that are relatively difficult to be mentally executed (P10: 75x36, 53.8% and P9: 28x56, 67.5%). The problems of medium difficulty demanded a sum of simple fractions with an addend being the ½ (P8: 1/2 +3/8, 72.5%), a calculation of a percentage close to 10% (P7: 9.84% of 816, 73.8%) which is very difficult to calculate with mental algorithm and the finding of an average of 6 measurements of Mount Everest (P6, 77.5%). Finally, teachers were more accurate in problems demanding an addition of 5 terms (P5: 35 +42 +40 +38 +44, 85%), a division (P4: 25,889 ÷ 52, 86.2%), a sum of six terms (P3: 27 +49 +38 +65 +56 +81, 91.3%), a sum of three terms (P2: 1,378 +236 +442, 93.8%) and a sum of decimal numbers (P1: 1.26 +4.79 +0.99 +1.37 +2.58, 93.8%). In other words, teachers were more (greater than 85%) accurate in problems demanding addition operations (i.e. P1, P2, P3 and P5) and division (i.e. P4), while they were less accurate (less than 68%) and had most difficulties in problems demanding multiplication (i.e. P9 and P10). For example, in problem P10 (75x36), some teachers did the accurate operation 3x7=21 → (30x70=2,100), but they found wrong the post compensation. Some of them did the operation 5x6=30 → 300 and resulted 2,100 + 300 = 2,400 and others did 5x6 = 30 and finally resulted 2,100 + 30 = 2,130. Many teachers, in this problem, didn’t make post compensation at all. Moreover, regarding the “type of number” involved in the computational estimation, the problem containing an addition of fractions (i.e. P8) was more difficult than the problems containing sums of natural and decimal numbers (i.e. P1, P2, P3 and P5).

**Strategies used in computational estimations**

**Table 2:** Percentages of strategies are used in accurate answers

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Front-end</th>
<th>Rounding</th>
<th>Special numbers</th>
<th>Clustering or Averaging</th>
<th>Compatible numbers</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>1+4+1+1+2</td>
<td>75x36=80x</td>
<td>⅜ is a half,</td>
<td>35+42+40+38</td>
<td>27+81=100</td>
<td></td>
</tr>
<tr>
<td></td>
<td>=9</td>
<td>40=3,200</td>
<td>3/8 is less</td>
<td>+44 all are</td>
<td>38+65=100</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.30+0.80+</td>
<td>with</td>
<td>than a half</td>
<td>close to 40</td>
<td>49+56=100</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.40+0.60=</td>
<td>compensat</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>ion</td>
<td>so 9+2=11</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
ELEMENTARY TEACHERS’ EFFICIENCY IN COMPUTATIONAL ESTIMATION PROBLEMS

Charalambos Lemonidis, Anastasia Mouratoglou, Dimitris Pnevmatikos

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Front-end</th>
<th>Rounding</th>
<th>Special numbers</th>
<th>Clustering or Averaging</th>
<th>Compatible numbers</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>30%*</td>
<td>55%</td>
<td></td>
<td></td>
<td>7.5%</td>
<td>1.3%</td>
</tr>
<tr>
<td>P2</td>
<td>11.2%*</td>
<td>80%</td>
<td></td>
<td></td>
<td>1.3%</td>
<td>1.3%</td>
</tr>
<tr>
<td>P3</td>
<td>2.5%</td>
<td>66.3%</td>
<td></td>
<td></td>
<td>22.5%*</td>
<td></td>
</tr>
<tr>
<td>P4</td>
<td>2.5%</td>
<td>30%</td>
<td></td>
<td></td>
<td>53.7%*</td>
<td>2.5%</td>
</tr>
<tr>
<td>P5</td>
<td>2.5%</td>
<td>28.7%</td>
<td></td>
<td></td>
<td>47.5%*</td>
<td>5%</td>
</tr>
<tr>
<td>P6</td>
<td>17.5%</td>
<td></td>
<td></td>
<td></td>
<td>52.5%*</td>
<td>1.3%</td>
</tr>
<tr>
<td>P7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>73.8%*</td>
<td></td>
</tr>
<tr>
<td>P8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>71.2%*</td>
<td></td>
</tr>
<tr>
<td>P9</td>
<td>5%</td>
<td></td>
<td>57.5%*</td>
<td></td>
<td></td>
<td>3.7%</td>
</tr>
<tr>
<td>P10</td>
<td>10%</td>
<td></td>
<td>42.5%*</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Percentages in bold indicate the appropriate strategy for each problem, according to the literature review.

Although the most appropriate strategy for problems P1 and P2 is considered to be in accordance with the previous research about the front end strategy, only 30% of teachers used this strategy properly in P1 and only 11.2% of teachers in P2 while the majority of the participants used the strategy of rounding (P1: 55% and P2: 80%). For example, in problem P1 (1.26 € + 4.79 € + 0.99 € + 1.37 € + 2.58 €), the majority of teachers rounded the numbers of the problem like 1.26→1.3, 4.79→4.8, 0.99→1, 1.37→1.4, 2.58→2.6 and then added them 1.3+4.8+1.4+2.6=11. So, they conclude that the sum of numbers was approximately 11. Similarly, although the most proper strategy for problems P3 and P4 is considered to be the compatible numbers strategy, this strategy was used only by 22.5% of teachers in the problem P3, while in the problem P4 by 53.7%. More specifically, a correct answer in problem 3 (27+49+38+65+56+81) would be 27+81=100, 38+65=100, 49+56=100, so the sum of the numbers would be approximately 300 (100+100+100). Instead, in the problem P3, the majority of teachers (66.3%) used the strategy of rounding. The strategy of accumulation or averaging is appropriate for the problems P5 and P6. The 47.5% of teachers used this strategy properly in the problem P5, and 52.5% of teachers used it correctly in problem P6. For example, in problem 5 (35+42+40+38+44≈200?), most of the teachers understood that all of the numbers were close to 40, so they estimated 5x40=200 and they concluded that the sum was approximately 200. The majority of teachers used the appropriate special numbers strategy to solve problems P7 and P8 (73.8% and 71.2% for the P7 and P8 problem respectively). For example, in problem P8 (1/2 km + 3/8 km), some teachers explained that ½=0.5 and 3/8<0.5 because 0.5=4/8, so the sum of fractions wasn’t above 1km. The most appropriate strategy for problems P9 and P10 namely the rounding strategy was used by the satisfactory rate of teachers (57.5% and 42.5% for P9 and P10 problem respectively). For example, in problem P10 (75x36), some teachers rounded up the number 36→40 and rounded down the number 76→70. After this, they multiplied them 40x70 and concluded in 2,800.
3.2. Factors related to personal characteristics of a solver and how they affect the teachers’ performance

In order to examine possible individual differences between the in-service teachers in their accuracy to computation estimation problems an aggregated score for each participant was created. Each accurate score was measured with one. Therefore, the scores ranged between zero and 10, with 10 indicating excellent accuracy. Overall, the average accurate responses were 7.95 with a standard deviation of 1.44. A series of t-tests was conducted to examine possible individual differences. In order to examine the impact of age in computation estimation accuracy in-service teachers were divided into two groups based on the median age (median = 46 years) of teachers. The two groups did not significantly differ in their accuracy, $t = 1.41, df = 78, p = .16$. Similarly, no significant difference, $t = .67, df = 78, p = .50$, is captured between male ($M = 7.82, SD = 1.44$) and female ($M = 8.04, SD = 1.44$) in-service teachers in their accuracy on computational estimation. In order to examine the impact of work experience in computation estimation accuracy in-service teachers were divided into two groups based on the median work experience age (median = 19.5 years) of teachers. The two groups did not significantly differ in their accuracy on computational estimation, $t = 1.09, df = 78, p = .280$. Moreover, in-service teachers did not significantly differ in terms of their experience with mathematics during the high school studies. Although teachers with mathematics as a major in their studies performed higher ($M = 8.08, SD = 1.46$) than the teachers who had attended courses in the humanities as a major ($M = 7.83, SD = 1.43$), this was not significant, $t = .760, df = 78, p = .450$. Participants were divided into two groups based on the median of the grades they had succeeded by the end of high school exams in mathematics (median = 18 out of 20). The analysis showed that the group with higher grades ($M = 8.24, SD = 1.44$) performed significantly higher, $t = 2.42, df = 78, p < .05$, than the teachers who were less effective ($M = 7.45, SD = 1.32$) in their mathematics’ exams at the end of high school.

Additionally, the participants were divided into two groups based on their emotional relationship to mathematics. Teachers who denoted good and very good emotional relationship with mathematics were classified as teachers with positive relationship (n=58), while the others were classified as teachers with negative emotional relationship with mathematics (n=22). The analysis showed that the group with the positive emotional relationship with mathematics ($M = 8.21, SD = 1.44$) performed significantly higher, $t = 2.69, df = 78, p < .01$, than the teachers who denoted a negative emotional relationship with mathematics ($M = 7.27, SD = 1.24$).

In sum, the analyses showed that the grade in mathematics’ exams at the end of high school, as well as their emotional relationships with mathematics are responsible for individual differences in teachers’ accuracy in computational estimation problems.
4. DISCUSSION

In this paper, we examined Greek in-service teachers’ accuracy in computational estimation and the strategies they use in their estimations. Moreover, we’re investigating possible individual differences in their accuracy. Seven are the main results of this study.

First, in-service teachers’ efficiency in computational estimation problems is not excellent, although they ought to address similar computational problems in their classes.

Second, in-service teachers overuse the rounding strategy. This is in accordance with previous studies concluding that the majority of teachers approach their computational estimation using the rounding strategy (e.g. Alajmi, 2009).

Third, in-service teachers rarely use and know a few about front-end strategy and compatible numbers strategy. This finding is partly consistent with the research of Lemonidis and Kaimakami (2013) which showed that pre-service teachers lack experience in using averaging and compatible numbers strategy.

Fourth, a significant percentage of teachers (around 70%) used the special numbers strategy. This strategy is an invented strategy and seems that teachers developed it on their own since the computational estimation strategies are not mentioned in the programs or textbooks.

Fifth, important evidence revealed about the role of arithmetic operations in computational estimations. The estimation problems containing the operation of addition are easier than problems containing the multiplication operation. Other investigations have similar conclusion too, such as investigation of Tsao (2013) and Lemonidis & Kaimakami (2013) in pre-service teachers who find that the addition problems are easier than the multiplication and division problems.

Sixth, teachers’ efficiency depends on the type of the number that is involved in the computational estimation. The sum of the fractions is much more difficult than the sum of decimals and whole numbers, in computational estimation problems. Containing an addition of fractions (1/2+3/8), problem P8 was more difficult than problems P1, P2, P3 and P5, which contained sums of natural and decimal numbers. This conclusion is in agreement with previous studies investigating the phenomenon in different samples such as pre-service teachers (Tsao, 2013), students of the fifth grade (Tsao & Pan, 2011) and high school students (Bana & Dolma, 2004).

Seventh, the study revealed two factors that create individual differences. These are the grade in mathematics final exam at high school and the in-service teachers’ emotional relationship with mathematics. Similarly, Tsao (2013) in his research on pre-service teachers found that good level of numeracy was associated with a good performance in computational estimation. Moreover,
positive emotionality towards mathematics was associated with good performance in computational estimation. Tsao (2013) had gone even further; he compared performance on computational estimations of pre-service teachers and their attitudes towards computational estimation. He found that there was a significant positive correlation between the experience with computational estimation, the confidence of computational estimation, the acceptability of computational estimation value, the fun of studying computational estimation, and the CET (Computational Estimation Test) scores.

However, the other possible for individual difference factors did not affect teachers’ accuracy on computational estimation. More specifically, their age group and work experience, the sex and the prior experience with mathematics did not affect teachers’ performance in computational estimation. This means that in-service teachers’ performance on computational estimation, doesn’t improve over time in their profession, and the way students approach mathematics in Greek high schools does not develop further computational estimations. For instance, Tsao and Pan (2013) found that teachers with mathematics / science major were relatively more abundant of the knowledge about computational estimation.

5. CONCLUSION

In-service Greek teachers use efficiently the computational estimations but there is still a space for improvement, as they face some difficulties with computational estimations demanding division and multiplication or rational numbers. Moreover, as they use a limited number of the existed strategies, there is more space for improvement in using a greater range of strategies. Finally, among the possible factors that were examined here, the emotional relationship with mathematics and their efficiency in the final exams at high school were important factors for individual differences in computational estimations. These conclusions, however, should be further explored in a future study, examining more teachers and using a thinking aloud approach to bigger number of participants.

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**BRIEF BIOGRAPHIES**

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EXPLORING NUMBER SENSE IN SIXTH GRADE IN GREECE: AN INSTRUCTIONAL PROPOSAL AND ITS LEARNING RESULTS

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ABSTRACT

Surveys show that the majority of Greek students use procedural strategies in regards to mental calculations with rational numbers, i.e., they are just trying to apply rules without having the adequate understanding (Lemonidis, 2013; Lemonidis & Kaiafa in this issue). This study proposes a teaching intervention which aims to advance students understanding about mental calculations and enrich their conceptual strategies repertoire. For this purpose, three teachers of 6th grade were first trained and, later, asked to teach mental calculations operations with rational numbers for a period of three months. The post training understanding and skills of 66 students were compared with the answers of 462 5th and 6th grade students who participated in mathematics competition by using the same instrument (Lemonidis and Kaiafa in this issue). The results showed that students of the experimental group, exploited a larger pool of strategies and, especially, more conceptual strategies. It is interesting that teachers were not familiar enough with the strategies for executing mental operations with rational numbers. After the intervention, positive changes in their attitudes and knowledge were identified.

Keywords: number sense improvement, rational numbers, sixth grader’s strategies.

1. INTRODUCTION

The teaching and learning of number sense in elementary and middle years has been considered to be a significant topic in mathematics education (Anghileri, 2007; Lemonidis, 2013; NCTM, 2000; Verschaffel, Greer, & De Corte, 2007; Yang, 2005). Number sense plays a key role for understanding rational numbers and their operations. Overemphasis on written computation often hinders the children’s mathematical thinking and comprehension (Anghileri, 2007; Reys & Yang, 1998).
However, related research studies show that children in elementary and middle grade levels are lacking number sense in rational numbers (Callingham, & Watson, 2004; Lemonidis, 2013; Markovits & Sowder, 1994; Reys & Yang, 1998; Yang, 2005). Therefore, teaching and learning number sense should be highlighted in elementary and middle school mathematics classrooms.

The teaching of rational numbers in Greece is concentrated more on rule-based calculations and less on in-depth understanding. In upper classes of elementary school, while the curriculum emphasizes the value of mental calculations, there are not concrete teaching proposals for mental computations with rational numbers. Our aim was to explore the effectiveness of a teaching intervention focused on mental calculations of rational numbers on teachers and students of sixth grade.

More specifically, the research questions in this paper were the following:
- Which were the pedagogical content knowledge and the practical skills of the teachers about mental calculations with rational numbers before and after the teaching intervention?
- How did the teaching intervention change the strategies with rational numbers of the sixth grade students?

2. BACKGROUND

Reys et al. (1999), in their review, identified six components of number sense:
1. Understanding of the meaning and size of numbers.
2. Understanding and use of equivalent representations of numbers.
3. Understanding the meaning and effect of operations.
4. Understanding and use of equivalent expressions.
5. Flexible computing and counting strategies for mental computation, written computation, and calculator use.
6. Measurement benchmarks (p. 62)

Yang and colleagues have coded subjects’ strategies as Number sense-based and Rule-based (Yang, 2003, 2005, 2007; Yang et al., 2009). Their criterion for distinguishing a strategy as a number sense-based was whether one or more components of number sense are evident in the person’s solution process (Yang, 2003, 2005, 2007). Some examples of number sense strategies are
a. the conversion of a fraction or a percentage to a decimal before operating on them,

b. the schematic representation of fractions (Caney and Watson, 2003)

c. residual thinking (Behr, et. al. 1984) where the fraction with the smaller residual is the bigger fraction.

On the other hand, rule-based strategies are based on memorized rules which are not necessarily linked to deep conceptual understanding.

Research has shown that many students in the upper and middle elementary grades are poor in number sense (Markovits & Sowder, 1994; McIntosh, et al. 1997;
Reys & Yang, 1998; Reys, et al. 1999; Van den Heuvel-Panhuizen, 1996, 2001; Yoshikawa, 1994; Verschaffel, Greer & DeCorte, 2007 Yang, 2005; Yang & Huang, 2004; Yang et al., 2008). The students are able to perform computations on familiar tasks by using school-learned procedures, but they perform poorly and employ rule-based methods on non-routine tasks.

Yang et al. (2009) have also identified teachers’ lack of number sense as a cause to students’ deficiencies in number sense: “Thus, children’s lack of number sense may be partly due to their teachers’ lack of number sense as well as not knowing how to help students develop number sense.” p. 383. Yang and colleagues have also argued that preservice teachers in Taiwan exhibit poor number sense, relying heavily on standard written algorithms (Yang, 2007; Yang et al., 2009). Although the preservice teachers were more capable of answering correctly number sense test items than their middle-school counterparts, the majority of their answers were also obtained by written computations based on standard algorithms.

However, number sense can be improved through instruction. Relevant studies with both students (Markovits and Sowder, 1994; Yang, 2002, 2003; Yang et al. 2004) and teachers (Kaminski, 2002; Whitacre, 2012, 2014; Whitacre and Nickerson, 2006) have demonstrated that number sense can be improved through instruction. For example, Yang (2003) reported on a semester-long (about 4 and a half months) quasi-experimental study of two fifth grade classes in Taiwan. Number sense activities were conducted in the experimental class, while the control class followed the standard mathematics curriculum. The authors used the Number Sense Rating Scale (Hsu et al., 2001) and found that the scores of the experimental class increased by 44% after instruction, while the scores of the control class increased only by 10%. Interviews showed that students from the experimental class used a higher proportion of number sense-based strategies in post-instruction and retention interviews. Whitacre (2012), in his literature review concluded that: “(1) number sense can be improved through instruction, and (2) there is more to be learned regarding how number sense improves with instruction.” p. 34.

3. METHODOLOGY

3.1. Research method

We used the framework of Ball and colleagues’ for examining teachers’ mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008). In their framework, the authors discern the knowledge that teachers apply in the classroom in two domains: subject matter and pedagogical content knowledge. Subject Matter Knowledge includes three subcategories: 1) Common Content Knowledge which refers to the general knowledge of mathematics not specific to teaching. 2) Specialized Content Knowledge which refers to the mathematical knowledge required for the specific learning content e.g. the knowledge teachers need to explain patterns of students’ errors or decide whether a nonstandard approach may work. 3) The Horizon Content
Knowledge which refers to the “awareness of how Mathematical topics are related over the span of mathematics included in the curriculum” (p. 403). Pedagogical Content Knowledge includes three subcategories: 1) Knowledge of Content and Students which combines knowing about students and knowing about mathematics. 2) Knowledge of Content and Teaching which concentrates on the design of instruction for the specific content. 3) Knowledge of Content and Curriculum which refers to the acknowledgement of the ways mathematics are developed in a teacher’s curriculum.

In our research, we examined the subject matter and pedagogical content knowledge of the three teachers, more specifically, we examined their Specialized Content Knowledge, Knowledge of Content and Students, and Knowledge of Content and Teaching.

Teachers attended a training program and then applied the newly acquired skills and knowledge in their classrooms. An action research approach was used to explore teachers’ subject matter and pedagogical content knowledge toward mental calculations with rational numbers before and after the intervention.

3.2. Participants

The participants in this study were three teachers of 6th grade and their 66 students. Teachers in this study were identified through a process of convenience sampling. Three teachers of 6th grade responded that they were interested in participating in the study. The teachers taught for the first time, in sixth grade.

3.3. Research Design

The study was conducted in five stages.

In the first stage, the authors interviewed 3 teachers of 6th grade. The aim of the interview was to explore their prior subject matter and pedagogical content knowledge for mental calculations on rational numbers.

In the second stage, a training session was conducted with each teacher separately. The training session lasted for two hours, and teachers were also given a 27-page booklet containing additional training material. The training took place before the Christmas holidays so as the teachers could have enough time to study the educational material, look at the various resources and organize their instructional design.

In the third stage, teachers used their instructional plans in the classroom. This stage lasted for 3 months.

In the fourth stage, the researchers conducted again personal interviews with each teacher in order to explore changes in their subject matter and pedagogical content knowledge. In those interviews, teachers also gave their views on the learning results of their new teaching approach.

Finally, in the fifth stage, questionnaires containing four problems of mental calculations with rational numbers were given to the 66 students. The problems were
identical to the ones given to 462 students of fifth and sixth grade who participated in a contest of mathematics. These students were not taught mental calculation strategies with rational numbers and, thus, the number sense strategies they used were spontaneous and self-developed. The comparison of these two groups of students could inform us about the effectiveness of the proposed teaching intervention.

3.4. Key features of teachers’ individual training session

The training session focused on the following topics:

- **The concept of mental calculations and their difference to written algorithms.**
- The aim was to clarify the meaning of mental calculations and their relationship with written calculation algorithms.
- **The value of mental calculations with rational numbers in number sense and operations.**
- The trainers emphasized that mental calculations of rational numbers improve the understanding of the operations and rational numbers in contrast to what happens with written algorithms where the operations can be executed without understanding.
- **Students’ strategies and errors**
- Teachers should be familiar with the various strategies that students exploit when performing operations with rational numbers. An analytic presentation of students’ systematic errors, difficulties and misconceptions, according to research data was given.
- **Instructional methods and metacognitive processes**

The importance of using metacognitive processes during the teaching of mental calculations was highlighted, and teachers were asked, for example, to make students talk and explain how they thought. The significance of presenting and sharing all students’ strategies was also indicated as a mean to enrich the strategies employed by students.

3.5. Data collection

Two types of data were collected:

- a. semi-structured interviews before and after the training session and the instruction and
- b. a questionnaire with 4 problems on rational numbers, which was administered to students in the end of the corresponding instruction.

3.6. Interviews

Before the intervention, the 3 teachers were asked about the importance of teaching mental calculations and whether it should follow or precede the teaching of rule-based calculations, about how they teach mental computations and what resources
they use and whether the majority of students in a typical classroom is able to develop and apply mental calculations when solving problems with rational numbers. The teachers were also asked to indicate the strategies that students use (in general and on specific problems) and the errors that students make on fractions, decimals and percentages.

After the intervention, the teachers were asked whether and how the training session together with the teaching intervention changed their initial perceptions about the importance of mental calculations, how they changed the way that they taught mental calculations, how students’ behavior changed and if they were satisfied with the initial training session.

3.7 Questionnaire

The four questions, that students of both groups had to answer, were:

Q1: I calculate with my mind: 1- ¼. I use two ways to answer. Every time, I write the way I thought.

Q2: I calculate with my mind: 1/2:1/4. I use two ways to answer. Every time, I write the way I thought.

Q3: I compare the fractions 3/7 and 5/8. Which is greater? I use two ways to answer. Every time, I write the way I thought.

Q4: I find the 90% of 40. I use two ways to answer. Every time, I write the way I thought.

4. RESULTS

4.1. Teachers’ subject matter and pedagogical content knowledge

4.1.1 Before the intervention

All three teachers considered significant the teaching of mental calculations and pinpointed a number of advantages. More specifically, teacher 1 emphasized the conceptual understanding that results from practicing with mental calculations, teacher 2 focused on the usefulness of mental calculations with rational numbers in everyday life, while teacher 3 noted the effect of mental calculations in connecting mathematics with everyday life and the advantages of releasing students from the rule-based algorithms.

The teachers had different views on the importance of teaching mental calculations and whether it should follow or precede the teaching of rule-based calculations. Teacher 1 considered mental calculations as more significant and argued that teaching of mental calculations should precede the teaching of rule-based thinking. Teacher 2 also supported the same order, but she didn’t select one of the two as more significant. Teacher 3 argued that teaching of mental and written calculations was of equal importance, however, she proposed the introduction of written calculations first and the presentation of mental calculations later: “First you need the theory and after having understand it, we can show them the clever ways”.
Teacher 3, considered written algorithms as the "theory" and mental calculations as merely some "smart ways" of calculation.

The teachers took advantage of mental calculations in different ways. The teacher 1 said that she devotes most of her time to mental calculations and that she tries to visualize the problems. Her goal was to focus more on understanding and less on algorithms. Teacher 2 said that she teaches mental calculations once a week with her own examples which are not contained in the textbook. The teacher 3, claimed that she teaches mental calculations when she gives explanations with small numbers on rule-based algorithms. All three teachers used a variety of resources (from textbooks to internet resources) and examples (from every day to school life). Interestingly, all three teachers said that they do not use computers because of logistical and technical difficulties, although they argued that they do appreciate the usefulness of technology.

Teachers were asked to assess whether the majority of students in a typical classroom is able to apply mental calculations when solving problems with rational numbers. According to their answers, they considered mental calculations as a difficult topic and believed that only the best students could develop such skills.

Teachers were also asked to indicate the strategies that students use on fractions, decimals and percentages. Teacher 1 stated that she did not remember any specific strategy, the other two said that students work on decimals in a similar way to the one of the whole numbers. Teacher 2 also argued that: "In fractions, my students calculate the difference between the numerator and the denominator." Teacher 3 also stated that students convert opposite fractions to decimals in order to operate on them. From the teachers’ responses, it was quite obvious that they did not know much about their students’ strategies when operating with rational numbers.

When asked about the common errors that students make with fractions, decimals and percentages, the three teachers identified different issues and each reported only a small number of problems and misconceptions. Teacher 1 reported that students often do not convert opposite fractions to the same denominator and they add the numerators and the denominators. For decimal numbers, they said that students do not pay attention to the place value. Teacher 1 also pointed out that there is a difficulty when dividing decimals or fractions. Teacher 2 noted the following two errors: Students think that the unit fraction is the number 1 and they believe that 0.4 and 0.40 are different. Students also consider that between two decimal numbers such as 1.4 and 1.5 no other numbers exist. Teacher 3 indicated that there is a general difficulty in understanding fractions and that student’s think that the multiplication of numbers with decimals results to smaller numbers.

To examine whether teachers were familiar with their students’ mental strategies, we asked them the following question:

“Which are the strategies that students would use if they were asked to estimate mentally the following operations: a) 1-1 / 4, b) ½: 1/4, c) compare the fractions 3/7 and 5/8 and d) find 90% of 40”
For the first question the strategies proposed were:
1. Rule-based: convert 1 to 4 quarters and remove one quarter.
2. Representational: Consider a geometric figure with four parts from which they remove \( \frac{1}{4} \).
3. Convert the whole number in the base of 100. Students will think of 1 as 100, divide 100 into 4 parts and remove thereby one part.

For this first question teacher 1 replied the strategies 1 and 2, teacher 2 the strategies 2 and 3 and teacher 3 only the strategy 1.

Teacher 1 said that the second problem was very difficult and that students might have solved it only by applying rule-based thinking. Teacher 2 said that students "would have divided one by two and then divided the result with two". Teacher 3 said: "I do not know, I need to solve it first by myself." Hence, the first teacher proposed a rule-based solution while the other two seemed unable to solve the problem by themselves.

For the comparison of fractions 3/7 and 5/8, teachers proposed the following strategies:
1. Convert the two fractions to same denominator.
2. Convert the two fractions to decimals.
3. Compare the two fractions with the benchmark of \( \frac{1}{2} \).

In this question, teacher 1 replied the strategies 1 and 3, teacher 2 named a wrong criterion ("3 of the 7 is more than 5 of 8"), and teacher 3 the strategies 1 and 2.

In the fourth question, teacher 1 did not suggest any strategy, teacher 2 replied that students would have multiplied mentally the numbers (4 x 9) and teacher 3 replied that she didn’t know because she hadn’t taught about percentages yet.

Hence, we can conclude that teachers did not know the variety of strategies exploited by students, but they also demonstrated a lack of content knowledge since they were not able to answer some of the questions.

4.1.2. After the intervention

When asked if the training session together with the teaching intervention changed their initial perceptions about the importance of mental calculations, teachers 1 and 3 said that they had already appreciated the value of mental calculations. Only teacher 3 noted that "after the training, I saw practically that mental calculations, can help students." Teacher 2 argued that the whole activity helped her revise some initial views as mental calculations seemed to "sharpen students’ minds". The teachers stated that they were satisfied with the initial training session.

Teachers suggested that all students were actively involved in their teaching while the weak students needed more time in order to familiarize with mental computations. Teacher 1 stated: "Initially, the mental computations with rational numbers intrigued the best students. However, later, the others and even some weaker students become more engaged. As they started to understand more the
strategies, they felt more confident about their answers and participated more actively in the course". The three teachers argued that all of their students participated in the classroom activities, even the weakest. Teacher 2 added: "Some students understood [rational numbers] through mental calculations and afterwards the written operations seemed as simple processes for them. Others started to understand better when the calculations were written. When the examples were simple and similar to the ones I used, students used mental computation with success. When the examples became more complex, students preferred written algorithms." Teachers also noted that many students knew and used appropriate calculation strategies for different problems.

4.2. Students’ behaviors after interventional instruction

4.2.1 Strategies usage

The following table shows the percentages of rule-based and number sense strategies that were used by the students (group A) who participated in the study and students (group B) who participated in the competition.

| Table 1: Use of strategies for students who succeeded in one or two questions |
|----------------------------------|---|---|---|---|
| STRATEGY                        | Q1:1-1/4 | Q2:⅓:1/4 | Q3:3/7 & 5/8 | Q4:90% of 40 |
| N=63                            | N=59     | N=48     | N=61          |
| N=269                           | N=230    | N=210    | N=224         |

**Rule-based strategies**

- Group A: 25 (39.7%), 31 (52.5%), 17 (35.4%), 33 (54%)
- Group B: 180 (67%), 185 (80.5%), 122 (58%), 140 (62.5%)

**Number sense strategies**

- Group A: 38 (60.3%), 28 (47.5%), 31 (64.6%), 28 (45.9%)
- Group B: 89 (33%), 45 (19.5%), 88 (41.9%), 84 (37.5%)

z values: 4, 4.39, 2.84, 1.19

p values: <0.001, <0.001, <0.01, <0.05

The students of group A used a smaller percentage of rule-based strategies and a larger percentage of number sense strategies in comparison to the students of group B. The 2-sample z-test validated this observation by indicating that in three (Q1, Q2 and Q3) of the four questions, the difference was significant. The proposed instruction promoted number sense strategies when operating on rational numbers. Students of group A understood better the operations with rational numbers despite the fact that the students who participated in the contest probably had more positive attitudes towards mathematics.

Another effect was that the respective students used a greater variety of number sense strategies. According to Table 2, in the questions Q1, Q2 and Q4 the students of the competition used 2, 2 and 3 number sense strategies respectively, while the students of group A used 3, 4 and 4 number sense strategies.
Table 2: Types of number sense strategies used by students

<table>
<thead>
<tr>
<th>STRATEGY</th>
<th>Q1:1-1/4</th>
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<tbody>
<tr>
<td></td>
<td>N=63</td>
<td>N=269</td>
</tr>
<tr>
<td>Number sense strategies</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Group A</td>
<td>38 (60.3%)</td>
<td>89 (33%)</td>
</tr>
<tr>
<td>Group B</td>
<td>28 (47.5%)</td>
<td>45 (19.5%)</td>
</tr>
<tr>
<td>Converting a Fraction or a Percent to a Decimal</td>
<td></td>
<td></td>
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<tr>
<td>Group A</td>
<td>23 (36.5%)</td>
<td>82 (30.5%)</td>
</tr>
<tr>
<td>Group B</td>
<td>17 (29%)</td>
<td>38 (16.5%)</td>
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<tr>
<td>Representation</td>
<td></td>
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<tr>
<td>Group A</td>
<td>9 (14.3%)</td>
<td>7 (2.5%)</td>
</tr>
<tr>
<td>Group B</td>
<td>7 (12%)</td>
<td>18 (8.5%)</td>
</tr>
<tr>
<td>Mixed representation</td>
<td></td>
<td></td>
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<tr>
<td>Converting to a decimal</td>
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<tr>
<td>Group A</td>
<td>6 (9.5%)</td>
<td></td>
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<tr>
<td>Group B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Benchmarks to ½ or 10%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Group A</td>
<td>2 (3.4%)</td>
<td>7 (3%)</td>
</tr>
<tr>
<td>Group B</td>
<td>6 (12.5%)</td>
<td>7 (3.5%)</td>
</tr>
<tr>
<td>Mixed representation</td>
<td></td>
<td></td>
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<tr>
<td>Benchmarks to ½</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Group A</td>
<td>2 (3.4%)</td>
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<tr>
<td>Group B</td>
<td></td>
<td></td>
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<tr>
<td>Residual thinking</td>
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<tr>
<td>Group A</td>
<td>2 (4%)</td>
<td>7 (3.5%)</td>
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<tr>
<td>Group B</td>
<td></td>
<td></td>
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<tr>
<td>Benchmarks: reduction in unit</td>
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<td></td>
</tr>
<tr>
<td>Group A</td>
<td></td>
<td>12 (19.5%)</td>
</tr>
<tr>
<td>Group B</td>
<td></td>
<td>23 (10.3%)</td>
</tr>
<tr>
<td>Benchmarks: 10% of 40=4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Group A</td>
<td></td>
<td>6 (10%)</td>
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<tr>
<td>Group B</td>
<td></td>
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</tbody>
</table>

5. DISCUSSION

The initial interview with the teachers revealed that they confronted serious shortcomings in the content knowledge of mental calculations with rational numbers, what Ball and colleagues (2008) named as Common Content Knowledge. It was also obvious that they were not aware of the knowledge, attitudes and misconceptions of students in the specific subject matter, what Ball and colleagues (2008) named as Specialized Content Knowledge. The lack of knowledge on the mathematical content
of mental calculations with rational numbers have also been noted in other studies (Post et al. 1988; Cramer & Lesh 1988; Khoury & Zazkis, 1994). Hence, it is necessary for teacher education programs to enhance Common Content Knowledge and Specialized Content Knowledge on mental calculations with rational numbers.

The teaching intervention improved students and teachers’ knowledge of number sense strategies as indicated by the interviews and the questionnaires. Students who belonged to the experimental group used number sense strategies in comparison with the students that participated in the competition. We have to note that the students of the second group participated in the competition by their own willingness, and hence, we can claim that this group was a challenging one for comparison since, at least those students had a more advanced interest in mathematics.

A limitation of this research is the lack of systematic monitoring of teachers’ instructional planning and decisions during the three month teaching period. Although, the teachers described in the final interviews their approaches, a more analytic recording of their practices would probably reveal more interesting issues about each class separately. Future research could also exploit the theory of mathematics teacher noticing (Jacobs, Lamb, & Philipp, 2010; Sherin, Jacobs & Philipp, 2011) in order to examine changes in teachers decisions and knowledge.

REFERENCES


**BRIEF BIOGRAPHIES**

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