Έχεις μοι εἰπεῖν, ὢ Σίκκρατες, ἄρα διδακτὸν ἢ ἄρετή; ἢ οὐ διδακτὸν ἄλλοι ἁγκητῶν, ἢ οὔτε ἁγκητῶν οὔτε ἁγκητῶν οὔτε μαθητῶν, ἄλλα φύσει παραγίγνεται τὸς ἀνθρώπους ἢ ἄλλω τινὶ τρόπῳ

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EDITOR'S INTRODUCTORY NOTE

INTRODUCTION TO SPECIAL ISSUE

Behaviour of students, teachers and future teachers in mental calculation and estimation

We are happy to present the first Special Issue of our new journal “MENON: Journal for Educational Research” which was introduced in 2012. Research in Mathematics Education is a significant area of educational research, which is included in the topics of this journal.

“Behaviour of students, teachers and future teachers in mental calculation and estimation” has been chosen as the subject for this special issue on the ground of a number of reasons which are presented below.

Over the past decades, many studies have been conducted in the field of mental calculations and estimation and more precisely in relation to the definition of these concepts, the identification of the strategies used by various age groups, the relationship with other concepts, such as number sense, the procedural and conceptual understanding among others.

Many educational systems have updated the teaching of numbers and operations in mathematics, incorporating mental calculations and estimations in their elementary and middle education curricula.

Nowadays, it is considered timely to conduct research in the implementation of the teaching of mental calculation and estimation with whole and rational numbers as well as the recording of students’ behaviour and the training of pre-service and in-service teachers in these concepts.

During the last decade, researches on mental computation and estimation with rational numbers have been conducted in the Laboratory of “Nature and Life Mathematic” at the University of Western Macedonia, some of which are presented in this issue.

Most of the papers included in this issue, refer to mental calculations and estimations with rational numbers, a topic that is not very common in the literature and covers a wide age range including elementary school students, adults, as well as pre- and in-service teachers. The researches are presented according to the age range of the participants.

- In their study Peters, De Smedt, Torbeyns, Ghesquière, Verschaffel distinguish between two types of strategies for subtraction: (1) direct subtraction, and (2) subtraction by addition, and provide an overview of the results of 5 related studies using non-verbal methods to investigate the flexible use of these strategies in both single- and multi-digit subtraction. Adults, students and
elementary school students with mathematical learning disabilities have participated in this research.

- **Anestakis and Lemonidis** in their study, investigate the computational estimation ability of adult learners and implement a teaching intervention about computational estimation in a Junior High School for Adults. They suggest incorporating computational estimation into Second Chance Schools and into adult numeracy teaching practices in general.

- The two papers of **Lemonidis, Nolka, Nikolantonakis** and **Lemonidis, Kaiafa** examine the behaviour of 5th and 6th grade students in computational estimation and in mental calculations with rational numbers, respectively. In these studies, the relation between students’ performance in computational estimation and mental calculations with rational number and problem solving ability are also examined.

Four studies on this issue, refer to the prospective elementary teachers' behaviour in mental calculation and estimation.

- Anestakis and Desli examined 113 prospective primary school teachers’ views of computational estimation and its teaching in primary school. Results revealed that the majority of prospective teachers identified the importance of computational estimation for both daily life and school.

- In their research Kourkoulos and Chalepaki interviewed and examined through a test 69 pre-service teachers aiming to investigate the factors that contribute to their computational estimation ability. They found five factors that contribute to computational estimation, such as the mathematical background and the attitude towards mathematics.

- Lemonidis, Tsakiridou, Panou and Griva used interviews to examine the knowledge and the strategy use of 50 pre-service teachers in multiplication tables and their mental flexibility in two-digit multiplications by using the method choice / no-choice by Lemaire & Siegler, (1995). The results showed that prospective teachers are not flexible in two-digit multiplications and their flexibility to mentally calculate two-digit multiplications is associated with their knowledge in prep and their prep response time.

- Koleza and Koleli have used a test to study the mental computations and estimation strategies of 87 pre-service teachers. The data revealed that the prospective teachers’ number sense concerning rational numbers, basic concepts of the decimal system and elementary numerical properties was very weak.

- Lemonidis, Mouratoglou and Pnevmatikos studied 80 in-service teachers’ performance and strategies in computational estimation and individual
differences concerning their age and work experience, their attitude towards mathematics and their prior performance in mathematics during high school years.

- The last paper of Lemonidis, Kermeli and Palaigeorgiou propose a teaching intervention to sixth grade students in order to promote understanding and enrich their conceptual strategy repertoire to carry out mental calculations with rational numbers. At the same time, three teachers’ attitudes towards teaching mental computation with rational numbers, were examined.

Finally, I would like to thank all the researchers from Belgium and Greece who contributed with their papers in this thematic issue, the colleagues from the laboratory of "Nature and Life Mathematics", the reviewers of the papers and Elias Indos for the organizational and technical support in the journal.

The Editor of the first Special Issue of "MENON: Journal for Educational Research"

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USING ADDITION TO SOLVE SUBTRACTION PROBLEMS IN THE NUMBER DOMAIN UP TO 20 AND 100

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ABSTRACT

Over the past 4 decades, a lot of research has been done on the strategies that are used to mentally solve subtraction problems. In the present manuscript, 2 types of strategies are distinguished: (1) direct subtraction, in which the smaller number is directly subtracted from the larger one (e.g., 75 − 43 = 32), and (2) subtraction by addition, in which one determines how much needs to be added to the smaller number to get to the larger one (e.g., 75 − 43 = 32). We set up a series of 5 related studies using non-verbal methods to investigate the flexible use of these strategies in both single- and multi-digit subtraction, first in adults and then in both typically developing children and children with mathematical learning disabilities. The present article provides an overview of the results of these 5 studies.

Keywords: mental arithmetic, direct subtraction, subtraction by addition, flexible strategy use.

1. INTRODUCTION

In the last four decades, a worldwide reform movement has changed some of the founding principles of elementary mathematics education. According to these
reform-based ideas, instruction should no longer focus on solving school mathematics exercises quickly and accurately by means of the school-taught standard strategies (i.e., routine expertise), but children should solve mathematical tasks efficiently, creatively, and flexibly with a variety of meaningfully acquired strategies (i.e., adaptive expertise) (e.g., Baroody & Dowker, 2003; Hatano, 2003; Kilpatrick, Swafford, & Findell, 2001; Verschaffel, Greer, & De Corte, 2007). Although this idea of flexibility is also endorsed in the current attainment targets of mathematics education in Flanders (Vlaams Ministerie van Onderwijs en Vorming, 2010), Flemish publishers of mathematics textbooks and elementary school teachers still seem to value the fast and accurate execution of one strategy over the flexible use of different (self-invented) strategies. Moreover, there is still a lot of discussion whether strategy variety and flexibility should also be seen as important goals for low-achieving children (e.g., Baroody, 2003; Kilpatrick et al., 2001; Threlfall, 2002; Verschaffel, Torbeys, De Smedt, Luwel, & Van Dooren, 2007).

One of the mathematical subdomains in which strategy variety and flexibility can be especially aimed for and stimulated, is mental subtraction. The way children and adults solve symbolically presented subtraction problems of the type \( M - S = \) (with \( M = \) minuend and \( S = \) subtrahend) has consequently received a lot of research attention (e.g., Barrouillet, Mignon, & Thevenot, 2008; Fuson, 1992; Kraemer, 2009; Robinson, 2001; Selter, 2001; Torbeys, De Smedt, Ghesquière, & Verschaffel, 2009a; Woods, Resnick, & Groen, 1975). These studies have shown that people develop various strategies to mentally solve subtraction problems. While researchers assumed for a very long time that for solving single-digit subtraction problems (e.g., \( 9 - 2 = \) or \( 13 - 7 = \)) children gradually move from counting-based strategies over procedural strategies to the (adult way of) direct retrieval of an answer from long-term memory, various studies from the last two decades showed that even adults often use non-retrieval strategies to solve subtraction problems (Ashcraft, 1992; Campbell & Xue, 2001; Geary, Frensch, & Wiley, 1993; LeFevre, DeStefano, Penner-Wilger, & Daley, 2006; Robinson, 2001; Seyler, Kirk, & Ashcraft, 2003). While retrieval seems to be mainly used for small single-digit subtraction problems (with minuend \( \leq 10 \)), large single-digit subtraction problems (with minuend between 10 and 20) are often solved by non-retrieval strategies based on counting (e.g., solving \( 11 - 2 = \) by counting down 2 starting from 11: “11, 10, 9”) or on the use of derived facts (e.g., solving \( 14 - 6 = \) by doing \( 14 - (4 + 2) = 10 - 2 = 8 \)).

In the domain of multi-digit subtraction, it has also been shown that adults and children develop various strategies to solve subtraction problems (e.g., Beishuizen, 1993; Blöte, Klein, & Beishuizen, 2000; Carpenter, Franke, Jacobs, Fennema, & Empson, 1997; Heirdsfield & Cooper, 2004; Selter, 1998; Thompson, 2000; Verschaffel, Greer, et al., 2007). Most of these studies categorised strategies into three main types, based on the way the subtraction operation is executed. In the first type, the split or decomposition strategies, the tens and the units of both minuend and subtrahend are split, and then these tens and units are subtracted separately.
from each other (e.g., $75 - 43 = . \text{ by doing } 70 - 40$ and $5 - 3$), so the answer is $30 + 2 = 32$). In the second type, the jump or sequential strategies, the minuend is kept as a whole and a problem is solved by subtracting the tens and the units of the subtrahend in two distinctive steps from the minuend (e.g., $75 - 43 = . \text{ by doing } 75 - 40 = 35$, and $35 - 3 = 32$). The third type of subtraction strategies, varying strategies, involves the flexible adaptation of the numbers and/or operations in the problem based on people’s understanding of the number relations or the properties of the arithmetic operations. An example of such a varying strategy is the compensation strategy, in which one or both of the numbers are changed to make the computations easier (e.g., $75 - 43 = . \text{ by doing } 75 - 45 = 30$, and then adding 2 because of the compensation of the original subtrahend, so the answer is $30 + 2 = 32$).

Most interesting within the context of the present manuscript is the observation that in both number domains (i.e., single-digit and multi-digit subtraction) adults and children sometimes solve subtraction problems by using an addition operation (Baroody, Torbeyns, & Verschaffel, 2009; Campbell, 2008; Fuson & Willis, 1988; Geary et al., 1993; Kraemer, 2009; Menne, 2001; Seyler et al., 2003; Torbeyns, De Smedt, Ghesquière, et al., 2009a; Torbeyns, Ghesquière, & Verschaffel, 2009; Verschaffel, Bryant, & Torbeyns, 2012; Woods et al., 1975). They report, for instance, that they know the result of a problem such as $7 - 3 = . \text{ because } 3 + 4 = 7$, they sometimes count on from the smaller to the larger number (e.g., solving $81 - 79 = . \text{ by doing } 79,... 80, 81, \text{ so the answer is } 2$), or they report that they solve a problem such as $75 - 43 = . \text{ by asking themselves how much needs to be added to the smallest number to get to the largest one, for example by doing } 43 + 2 = 45$ and $45 + 30 = 75$, so the answer is $2 + 30 = 32$. Consequently, a second type of classification of subtraction strategies is possible, by looking at the operation that underlies the solution process. In this way, two types of strategies can be distinguished: (1) direct subtraction strategies, in which the subtrahend is directly subtracted from the minuend (e.g., $75 - 43 = . \text{ by } 75 - 40 = 35$, $35 - 3 = 32$), and (2) subtraction by addition strategies, in which one determines how much needs to be added to the subtrahend to get to the minuend (e.g., $75 - 43 = . \text{ by } 43 + 30 = 73$ and $73 + 2 = 75$, so the answer is $30 + 2 = 32$).

Over the years, several studies at the Leuven Centre for Instructional Psychology and Technology (CIP&T) have investigated the use of the subtraction by addition strategy on symbolically presented subtraction problems. They used Lemaire and Siegler’s (1995) model of strategy change and strategy choice as a theoretical framework, and particularly focused on the parameter that refers to the adaptiveness or flexibility of someone’s strategy choices. In this model of strategy change and strategy choice, someone’s strategy choice is called flexible if (s)he chooses the strategy from his/her strategy repertoire that will lead fastest to an accurate answer. Through practicing several similar problems over time, someone will learn to use more efficient strategies, which will be executed more frequently, more efficiently, and also more adaptively.
Direct subtraction and subtraction by addition are two strategies for which an adaptive strategy choice based on task characteristics\(^3\) can be particularly efficient (e.g., Torbeyns, De Smedt, Stassens, Ghesquière, & Verschaffel, 2009): Based on a rational task analysis, direct subtraction is assumed to elicit few and/or small counting/calculation steps when the subtrahend is relatively small compared to the difference (such as \(9 - 2 = ., 12 - 3 = .\) or \(81 - 2 = .\)), but more and/or larger steps when the subtrahend is relatively large compared to the difference (such as \(9 - 7 = ., 12 - 9 = .\) or \(81 - 79 = .\)). Following the same logic, the opposite process is expected to happen for the subtraction by addition strategy: Few and/or small counting/calculation steps are needed when the subtrahend is relatively large (such as \(9 - 7 = ., 12 - 9 = .\) or \(81 - 79 = .\)), but more and/or larger steps when the subtrahend is relatively small compared to the difference (such as \(9 - 2 = ., 12 - 3 = .\) or \(81 - 2 = .\)). Selecting the strategy for which the counting/calculation steps are very small and easy can be seen as particularly efficient, because this quick and easy counting/subtraction process will very often also lead to a correct answer. Compare, for example, solving \(81 - 79 = .\) by means of direct subtraction (e.g., \(81 - 70 = 11\) and \(11 - 9 = 10 - 8 = 2\)) with using the subtraction by addition strategy to solve it (e.g., \(79 + 1 = 80\) and \(80 + 1 = 81\), so the answer is \(1 + 1 = 2\)).

While these previous studies have shown that adults use the subtraction by addition strategy frequently, efficiently, and flexibly (i.e., mainly, but not exclusively, on problems with a relatively large subtrahend) (e.g., Torbeyns, Ghesquière, et al., 2009; Torbeyns, De Smedt, Peters, Ghesquière, & Verschaffel, 2011), well-documented evidence of elementary school children’s use of the subtraction by addition strategy is very scarce (e.g., De Smedt, Torbeyns, Stassens, Ghesquière, & Verschaffel; 2010; Torbeyns, De Smedt, Ghesquière, et al., 2009a, 2009b). For example, Torbeyns, De Smedt, Ghesquière, et al. (2009a) asked Flemish second-, third-, and fourth-graders to solve two-digit subtractions in two tasks. In the Spontaneous Strategy Use Task, children were asked to solve each problem as fast and as accurately as possible with their preferred strategy, and to verbally report both the answer and the strategy they used after solving each problem. Five out of the 15 presented two-digit subtraction problems had a relatively large subtrahend (as in \(81 - 79 = .\)), which was assumed to trigger the subtraction by addition strategy. Still, children hardly applied this strategy spontaneously: Second- and third-graders did so in about 5% of the cases on the five problems with a relatively large subtrahend; fourth-graders in only 9% of these cases. In the Variability on Demand Task, children were asked to generate up to five different strategies for solving 4 two-digit subtraction problems. Now, two of these problems were assumed to trigger the subtraction by addition strategy. Surprisingly, the frequency of subtraction by addition strategies did not differ that much from the first task: Only 4% of the second-graders, 13% of the third-graders, and 21% of the fourth-graders reported subtraction by addition as a possible way to solve the two problems. The authors
concluded that the subtraction by addition strategy was not included in most children’s strategy repertoire.

Similar results were reported by Torbeys, De Smedt, Ghesquière, et al. (2009b) when comparing two groups of second- to fourth-graders who were asked to write down their solution steps when solving symbolically presented two-digit subtraction problems. These two groups only differed in the received instruction about the subtraction by addition strategy. In one group no instruction about this strategy was given, whereas the second group had followed a mathematics textbook from first grade on that focused on subtraction by addition as the alternative for the direct subtraction strategy for problems with a relatively small difference⁴. Both groups of children were asked to solve the same 16 symbolically presented two-digit subtractions in whatever way they wanted, and to write down their solution steps. Half of the experimental items were designed with a difference smaller than 10, in order to elicit the use of subtraction by addition as much as possible. However, only 0.2 % of all written reports in the no-instruction group, but also no more than 7.5 % of all written solutions in the instruction-group, could be identified as subtraction by addition. Again, the authors concluded that the subtraction by addition strategy was hardly used, even by children who had received instruction about and practice in this strategy.

De Smedt et al. (2010) tried to stimulate the use of subtraction by addition in third-grade children. Participants were divided over an implicit and explicit learning environment, both of which involved four training sessions. In the implicit learning environment, children were confronted with an unusually large number of two-digit subtraction problems with a relatively large subtrahend. This was done because Flemish textbooks hardly contain this type of subtraction problems, although they are most suitable for discovering the computational advantage of the subtraction by addition strategy. Children in the explicit learning environment were instructed to solve each problem once with direct subtraction and once with subtraction by addition, which was explained at the beginning of each training session. None of the children from the implicit learning environment reported using the subtraction by addition strategy in the test session halfway the training, at the end of the training, or in the retention session one month later. In the explicit learning environment, only 6 % of the children reported using the subtraction by addition strategy in the test session after two training sessions, only 11 % reported subtraction by addition by the end of the training sessions, and only 10 % reported it one month later. From these low percentages, the authors inferred that – even in the explicit learning environment – third-grade children experienced great difficulties with picking up and integrating the subtraction by addition strategy into their strategy repertoire.

2. FIVE RELATED STUDIES ON THE FLEXIBLE USE OF THE SUBTRACTION BY ADDITION STRATEGY

One important limitation of the studies reviewed above is that they relied exclusively on verbal or written data to detect the subtraction by addition strategy. These
methods might, however, not be the best way to identify certain types of mental calculation strategies (e.g., Cooney & Ladd, 1992; Ericsson & Simon, 1993; Kirk & Ashcraft, 2001; LeFevre, Smith-Chant, Hiscock, Daley, & Morris, 2003; LeFevre, Sadesky, & Bisanz, 1996; Russo, Johnson, & Stephens, 1989; Siegler & Stern, 1998). Kirk and Ashcraft (2001), for example, wrote that the validity of using verbal protocols can be questioned when studying certain mathematical processes, a statement they base on problems with veridicality, reactivity, and demand. First, they argue that the mental processes that underlie a solution process cannot always be accurately reflected on in verbal reports, especially when the execution of the solution process is automatic (i.e., not involving working memory processing). When solving a problem such as $12 - 9 = . \text{ or } 81 - 79 = . \text{ (i.e., problems for which the subtraction by addition strategy seems to be very efficient), people may thus not be aware of the calculation steps they executed, and only have access to the outcome of the problem. Strategy reports for these problems might therefore not be veridical to what really happened when solving the problem. Secondly, according to Kirk and Ashcraft, asking participants how they solve a problem might change the mental processes occurring in normal settings (see also Ericsson & Simon [1993] and Russo et al. [1989]): It might be that participants are pressured to perform better in terms of accuracy since verbal strategy reports seem to be a platform that explicitly shows the errors they make, which forces them into changing their normal routines. Similarly, they might even deliberately change the strategies they would spontaneously use because they know they might not be able to explain them and therefore choose to select strategies from their strategy repertoire which they do have the words for. In other words, they might react to the setting that they are put in when participating in an experiment.

As a third problem, Kirk and Ashcraft discuss the possibility that participants might figure out the goal of an experiment and therefore state answers or processes that they think the experimenters are interested most in. Participants might thus deliberately hide the use of a particular strategy because they think it is not valued or even not allowed, and therefore report the strategies they think the setting demands. Especially in children - who are in their classroom often confronted with only one strategy to solve certain types of problems and therefore might have the idea that other strategies are not allowed (socio-mathematical classroom norms, Yackel & Cobb [1996]) - this might be a very plausible reaction.

When taking these possible methodological problems into account, the results of previous research on children’s use of the subtraction by addition strategy might represent an underestimation of their use of this strategy. In this respect, we point to an inconsistency between the verbal reports and children’s reaction time data in De Smedt et al. (2010). The vast majority of the third-graders in this study only reported to use the direct subtraction strategy. However, if this had actually been the case, there should have been an increase in children’s reaction times from problems with relatively small subtrahends (e.g., $81 - 7 = . \text{) over problems with medium-sized subtrahends (e.g., } 81 - 43 = . \text{) to problems with relatively large subtrahends (e.g., } 81$
− 79 = .), because according to the above-mentioned rational task analysis, subtracting a larger subtrahend requires more and/or larger calculation steps. The observed reaction time patterns, however, argue against this interpretation: not only problems with relatively small but also problems with relatively large subtrahends were solved faster than problems with medium-sized subtrahends. These reaction time data thus suggest that the verbal report data were not always in line with the strategies actually applied by these children. More specifically, they indicate that the subtraction by addition strategy might have been used more frequently than suggested by the children’s verbal reports.

We therefore aimed at studying the use of subtraction by addition with other, non-verbal methods for inferring strategy use. More particularly, we applied non-verbal methods for investigating the flexible use of subtraction by addition in both single- and multi-digit subtraction, first in adults and then in both typically developing children and children with mathematical learning disabilities. Here we will present a short overview of all five studies; for more details we refer to the original manuscripts (Peters, 2013; Peters, De Smedt, Torbeyns, Ghesquière, & Verschaffel, 2010a, 2010b, 2012, 2013; Peters, De Smedt, Torbeyns, Verschaffel & Ghesquière, 2014).

3. STUDY 1

In Study 1 (Peters et al., 2010a), we investigated 25 university students’ (Mean age = 26 years; SD = 7 years) use of the subtraction by addition strategy when solving large single-digit subtraction problems (i.e., problems with a minuend larger than 10 and a single-digit subtrahend). We extended the work of Campbell (2008), who presented students with subtraction problems in the standard subtraction format (9 − 2 =.) and in their corresponding addition format (2 + . = 9) and who studied the effect of these different presentation formats on adults’ reaction times. Campbell found that problems presented in the addition format were solved faster than those in subtraction format and he concluded that large single-digit subtractions are often solved by means of addition.

However, previous research on related arithmetic tasks (e.g., Brissiaud, 1994; Torbeyns, Ghesquière et al., 2009; Woods et al., 1975) suggests that this use of subtraction by addition may depend on the relative size of the subtrahend, a task parameter that was not systematically addressed in Campbell (2008). We therefore extended Campbell’s research method and tested whether students solved large single-digit subtractions by flexibly switching between the direct subtraction and the subtraction by addition strategy, depending on which of both processes requires the fewest or smallest steps. We compared solution times on the 32 large non-tie subtraction problems presented once in the standard subtraction format (12 − 9 = .) and once in an addition format (9 + . = 12). We systematically manipulated the relative size of the subtrahend in these problems by combining two problem characteristics: the magnitude of the subtrahend (S) compared to the difference (D) (S < D vs. S > D), and the numerical distance between S and D (small-distance
problems were defined by $S$ and $D$ differing by only 1 or 2, whereas in the large-distance problems $S$ and $D$ differ by more than 2).

All participants performed a computer task in which they had to solve a total of 64 items. Each trial started with an asterisk that appeared for 1000 ms in the centre of the screen. Next, the item was presented (horizontally) in the middle of the screen. Participants’ solution time started to run when the item appeared on the screen, and ended when a sound was detected. We performed a repeated measures ANOVA on the solutions times with magnitude of the subtrahend ($S < D$ vs. $S > D$), numerical distance (large vs. small), and format (subtraction vs. addition) as within-subject factors. We found a significant three-way interaction: When the subtrahend was larger than the difference and $S$ and $D$ were far from each other (e.g., $12 - 9 = .)$, problems were solved faster in the addition than in the subtraction format; when the subtrahend was smaller than the difference and $S$ and $D$ were far from each other (e.g., $12 - 3 = .)$, problems were solved faster in the subtraction than in the addition format. However, when the subtrahend and the difference were close to each other (e.g., $13 - 6 = .$ and $15 - 8 = .$), there were no significant reaction time differences between both formats. These results suggest that adults do not rely exclusively on addition to solve large single-digit subtractions, but select either direct subtraction or subtraction by addition, depending on the relative size of the subtrahend.

4. STUDY 2

In Study 2 (Peters et al., 2010b), we extended our research question to the number domain up to 100. Several authors reported adults using the subtraction by addition strategy in multi-digit subtraction. Starting from the results of Torbeyns, Ghesquière, et al. (2009), we hypothesised that we would again find a flexible strategy choice pattern between direct subtraction and subtraction by addition, based on the relative size of the subtrahend. Twenty-five university students (Mean age = 26 years; $SD = 7$ years) were asked to solve 32 two-digit subtraction problems, once presented in the standard subtraction format ($81 - 37 = .$) and once in an addition format ($37 + . = 81$). We first calculated a stepwise regression model – based on Groen and Poll (1973) and Woods et al. (1975) in the domain of single-digit arithmetic – in which participants’ reaction times on two-digit subtractions were predicted by the presentation format ($M - S = .$ or $S + . = M$), three variables referring to strategy use (the to-be-determined difference $[D]$, the known subtrahend $[S]$, and the minimum of difference and subtrahend $\min[D, S]$), and their respective interactions. We expected that the model including the min ($D, S$) predictor as the variable that explains most of the variability in reaction times would provide the best fit, suggesting the mixed use of subtraction by addition and direct subtraction in both presentation formats. This was indeed the case.

Second, we performed exactly the same analysis as in Study 1. The 32 two-digit problems were divided into four problem types, based on the combination of the magnitude of $S$ compared to $D$ ($S < D$ or $S > D$) and the numerical distance between $S$
and $D$ (small or large). Small-distance problems were defined by $S$ and $D$ differing by less than 10, whereas in the large-distance problems $S$ and $D$ differed by at least 10 and either $S$ or $D$ was a one-digit number. We again compared reaction times on the problems presented in the two presentation formats, and found that students switched between direct subtraction and subtraction by addition depending on the relative size of the subtrahend: If the subtrahend was smaller than the difference (e.g., $83 - 4 = .)$, direct subtraction was mainly used; if the subtrahend was larger than the difference (e.g., $83 - 79 = .$), subtraction by addition was the dominant strategy. However, this performance pattern was only observed when the distance between the subtrahend and the difference was large; when the subtrahend and the difference were close to each other (e.g., $81 - 37 = .$ or $81 - 44 = .$) there was no subtrahend-dependent selection of direct subtraction vs. subtraction by addition.

The results of Study 2 thus indicate again that the relative size of the subtrahend is an important factor in the strategy selection process.

5. STUDY 3

In Study 3 (Peters et al., 2012), we focussed on primary school children solving large single-digit subtraction problems (i.e., problems with a minuend larger than 10 and a single-digit subtrahend). Whereas several studies showed that children seem to apply the subtraction by addition strategy when confronted with small single-digit subtraction problems (with a minuend $\leq 10$) (e.g., Barrouillet et al., 2008; Carpenter & Moser, 1984; Fuson, 1992), there is hardly any research concerning children’s flexible use of the subtraction by addition strategy on large single-digit subtraction problems. Therefore, we tested this issue through a replication of our study on adults’ use of subtraction by addition on large single-digit subtraction (Study 1) in elementary school children, now using regression-based analyses.

We presented 106 third- to sixth-graders (mean ages were 8 years 11 months [$SD = 3$ months] in third grade, 9 years and 11 months [$SD = 4$ months] in fourth grade, 10 years and 10 months [$SD = 4$ months] in fifth grade, and 11 years and 10 months [$SD = 3$ months] in sixth grade) with the 32 non-tie subtraction problems, once in the standard subtraction format $(12 - 9 = .)$ and once the addition format $(9 + . = 12)$. For both presentation formats separately, we compared the fit of three linear regression models, which represented, respectively, the consistent use of direct subtraction, of subtraction by addition, and of flexibly switching between both strategies based on the relative size of the subtrahend. The first model, the $DS$-Model, represented children only using the direct subtraction strategy. If this was the case, their reaction times should be best predicted by the size of the known subtrahend ($S$), because it takes longer to subtract 9 from a given number than to subtract 3 from that number. In the second model, the $SBA$-Model, the consistent use of the subtraction by addition strategy was represented. According to this model, children’s reaction times should be best predicted by the size of the to-be-determined difference ($D$), because it takes longer to determine how much needs to be added to get at a given number.
when the difference between both numbers is relatively large (“How much needs to be added to 3 to have 12?”) than when it is relatively small (“How much needs to be added to 9 to have 12?”). Finally, if children switched flexibly between both strategies depending on which strategy is most efficient, as represented by the Switch-Model, reaction times should be best predicted by the minimum of the subtrahend and the difference (min(D, S)): For problems with the subtrahend smaller than the difference (e.g., 12 – 3 = . and 12 – 5 = .), we expect reaction times to increase with the size of the subtrahend, because these problems can be quickly solved by means of the direct subtraction strategy. In contrast, for problems with the difference smaller than the subtrahend (e.g., 12 – 9 = . and 12 – 7 = .), we expect reaction times to increase with the size of the difference, because these problems can be quickly solved by means of subtraction by addition. Findings revealed that children did not switch flexibly between the two strategies, as adults did in Study 1, but that they relied only on the direct subtraction strategy to solve the standard subtraction problems (M – S = .) and only on subtraction by addition for problems presented in the addition format (S + . = M), independently of the relative size of the subtrahend.

6. STUDY 4

In Study 4 (Peters et al., 2013), we replicated the study on adults’ use of the subtraction by addition strategy on two-digit subtraction problems (Study 2) in 72 fourth- to sixth-grade elementary school children (mean ages were 9 years and 9 months [SD = 3 months] in fourth grade, 10 years and 10 months [SD = 4 months] in fifth grade, and 11 years and 9 months [SD = 3 months] in sixth grade). We presented them with 32 two-digit subtractions, which could be classified into four problem types (see Study 2). First, we fitted the same three regression models as we did in Study 3 to the reaction times of these 32 two-digit subtractions. These models represented either the use of direct subtraction, subtraction by addition, and switching between the two strategies based on the magnitude of the subtrahend. Additionally, we compared reaction times on problems presented once in the standard subtraction format (81 – 37 = .) and once in an addition format (37 + . = 81), as we did in Studies 1 and 2. Both methods converged to the conclusion that children of all three grades switched between direct subtraction and subtraction by addition based on the combination of two features of the subtrahend: If the subtrahend was smaller than the difference (e.g., 83 – 4 = .), direct subtraction was the dominant strategy; if the subtrahend was larger than the difference (e.g., 83 – 79 = .), subtraction by addition was mainly used. However, this performance pattern was only observed when the numerical distance between subtrahend and difference was large.

7. STUDY 5

Finally, we investigated the use of subtraction by addition in children with mathematical learning disabilities (MLD) (Peters et al., 2014). As stated before,
especially for these children the idea of stimulating strategy variability and flexibility is still subject to discussion among scholars (e.g., Baroody, 2003; Kilpatrick et al., 2001; Threlfall, 2002; Verschaffel, Torbeyns, et al., 2007). Some researchers and policy makers advise to teach MLD children only one solution strategy, others advocate stimulating the flexible use of various strategies, as for typically developing children. To contribute to this debate, we investigated the use of the subtraction by addition strategy to mentally solve two-digit subtractions in 44 children with MLD (mean age 12 years and 5 months [SD = 6 months]). We conducted a replication of Study 4, and thus again used two non-verbal research methods to infer strategy use patterns. First, we fitted three regression models to the reaction times of 32 two-digit subtractions. Additionally, we compared performance on problems presented in two presentation formats (e.g., 81 – 37 = . and 37 + . = 81). We found that MLD children – similar to their typically developing peers – switch between the traditionally taught direct subtraction strategy and subtraction by addition, based on the relative size of the subtrahend. These findings challenge typical special education classroom practices, which only focus on the routine mastery of the direct subtraction strategy.

8. CONCLUSION AND DISCUSSION

We have reported on five closely related studies on the use of the subtraction by addition strategy to solve symbolically presented subtraction problems. In all studies, we investigated whether adults (Studies 1 and 2) or elementary school children (typically developing children in Studies 3 and 4, and children with MLD in Study 5) switch between the subtraction by addition strategy and the direct subtraction strategy to solve subtraction problems, and whether they base their strategy choice on the relative size of the subtrahend. Based on previous work in the fields of cognitive psychology and mathematics education (e.g., Torbeyns, Ghesquière, et al., 2009; Woods et al., 1975), we hypothesized that this task characteristic would influence the strategy choice process because of its connection to the efficiency of the calculation process: Problems with a relatively small subtrahend (such as 12 – 3 = . or 81 – 2 = .) can be solved very fast and easy by taking away the subtrahend from the minuend, whereas for problems with a relatively large subtrahend (such as 12 – 9 = . or 81 – 79 = .) it can be faster and easier to determine the difference by adding on to the subtrahend to get to the minuend.

We focused on strategy use on symbolically presented problems in two mathematical domains, i.e., large single-digit subtraction (with minuends between 10 and 20) and two-digit subtraction. In all five studies, all problems involved borrowing and were presented in both the standard subtraction format (M − S = .) and the addition format (S + . = M). In four of the five studies participants based their strategy choices on the relative size of the subtrahend: For both presentation formats, they showed strategy use patterns which represent the use of direct subtraction when the subtrahend was relatively small and subtraction by addition
when the subtrahend was relatively large. These patterns were found for adults who solved large single-digit and two-digit subtractions, and for both typically developing children and children with MLD who solved two-digit subtractions. However, a different result was found for the typically developing children who solved large single-digit subtraction problems such as $12 - 9 = ?$ in Study 3: Reaction time patterns suggested that these children used the direct subtraction strategy when those problems were presented in the standard subtraction format ($M - S = .), and subtraction by addition for problems presented in the addition format ($S + . = M$), independently of the relative size of the subtrahend.

How can we explain this latter contrasting finding? Since the results presented in Studies 4 and 5 showed that children flexibly switched between direct subtraction and subtraction by addition in the number domain up to 100, a lack of conceptual understanding of the addition/subtraction complement principle, the inability of inhibiting the taking-away interpretation of the minus sign, and the effect of classroom norms and practices through which flexible strategy use is not stimulated (i.e., three possible explanations given in Peters et al., 2012) seem to be invalidated. Two other explanations, namely (1) the way subtraction with borrowing at 10 is instructed in Flemish elementary schools and (2) the influence of a strategy switch cost (i.e., the two remaining explanations given in Peters et al., 2012) still seem valuable explanations for this contrastive finding. First, the instruction in solving large single-digit subtractions in Flemish elementary schools focuses primarily on splitting the subtrahend into two parts, in order to substitute the original problem into two easier problems involving the number 10 (Van Olmen, 2005). From first grade on, children practice this decomposition strategy so often that the confrontation with a large single-digit subtraction problem may automatically trigger the use of the direct subtraction strategy in that number domain. Second, and somewhat related to the first explanation, it might be that children do not switch between the two strategies because the possible advantages of such a switch do not compensate for the cognitive cost (in terms of time and effort) involved in making the strategy switch (e.g., Lemaire & Lecacheur, 2010; Luwel, Onghena, Torbeys, Schillemans, & Verschaffel, 2009). They might have been so proficient in using the direct subtraction strategy on problems in subtraction format that merely the process of switching to an alternative strategy would involve a cognitive cost being larger than sticking to direct subtraction. Both remaining explanations should be addressed in more detail in future research.

The finding that both typically developing children and children with MLD flexibly choose between the two strategies is in contrast with previous research on children’s strategy use on symbolically presented two-digit subtraction problems (e.g., De Smedt et al., 2010; Torbeys, De Smedt, Ghesquière, et al., 2009a, 2009b). As we stated in the introduction, in this previous research hardly any children reported to use the subtraction by addition strategy when they were asked to explain how they solved the problems. Based on the findings reported in the present overview, we
have to question why in previous research children did not report using the subtraction by addition strategy. Were they not aware of the calculation steps they had executed? Did they have difficulties in articulating precisely how they found the answer (and therefore reported the direct subtraction strategy, which they had learnt to verbalize during the numerous mathematics lessons wherein they had practiced that strategy)? Or, did they deliberately hide the use of the subtraction by addition strategy because they taught it was not valued, or even not allowed, to solve a subtraction problem in that way? All these explanations seem possible, and should be examined in more detail in further research. However, the two non-verbal research methods used in the five present studies have their limitations too (see Peters, 2013). Other research methods, such as using eye-movements, could therefore be used in future studies to further investigate and enhance the validity of both verbal and non-verbal research methods through triangulation.

At a more general theoretical level, our findings are in accordance with Siegler’s SCADS* model (Siegler & Araya, 2005; see also Verschaffel, Luwel et al., 2009). This model contends that when confronted with cognitive tasks people make adaptive strategy choices and take into account knowledge about the efficiency of a particular strategy for a particular problem type. Four of the five present studies revealed that both adults and children rely on two task features when choosing between the direct subtraction and the subtraction by addition strategy: (1) the magnitude of the subtrahend, and (2) the numerical distance between subtrahend and difference (the combination of which has been termed by Peters (2013) and also in this article as the relative size of the subtrahend). Taking into account these two task characteristics involves some kind of comparison between the numbers in the problem. Arguably, this comparison occurs during the orienting or planning phase of the solution process and relies on fast (quasi-)automatic processes of estimation and number sense rather than precise calculations of differences between the numbers. Several studies have highlighted the importance of magnitude comparison skills and number sense for successful mathematical development (e.g., De Smedt, Verschaffel, & Ghesquière, 2009; Gilmore, McCarthy, & Spelke, 2007; Holloway & Ansari, 2009; Sasanguie, Van den Bussche, & Reynvoet, 2012; Vanbinst, Ghesquière, & De Smedt, 2012). Future research should investigate the role of these characteristics in the execution of strategies in both single- and multi-digit subtraction in more detail. Based on the results presented in Study 3, the role of the presentation format and the avoidance of strategy switch cost should also be further investigated. All this suggests that in the future not only the influence of task characteristics on strategy flexibility should be looked into, but subject characteristics should be included as well. In that way, theoretical models about people’s strategy choices when solving symbolically presented subtraction problems can be further refined.

Finally, we discuss some implications of our results for mathematics education. First, we want to stress that additional questionnaires with teachers in Studies 3, 4, and 5 showed that in the majority of the participating schools the subtraction by
addition strategy was not systematically and explicitly taught as a valuable alternative for the direct subtraction strategy to solve symbolically presented subtraction problems. Still, according to our non-verbal methods, this strategy was part not only of typically developing but also of MLD children’s strategy repertoire. In the number domain up to 100 (presented in Studies 4 and 5) the subtraction by addition strategy was even used flexibly, based on the relative size of the subtrahend. Taking also into account that the children’s mathematical textbooks did not provide them with many opportunities to discover and practice this strategy – since the Flemish textbooks hardly contain problems with a relatively large subtrahend (De Smedt et al., 2010) – this result was quite surprising, and it shows that teachers should be aware that generally speaking children (both in normal and special education schools) can do more than they are taught to do and more than is expected from them.

Second, teachers, teacher trainers, and material developers should be made aware of the possible problems linked to asking children how they solved a problem, so they can pay attention to them in their practices. Taking into account the attainment targets for Flemish elementary mathematics education (Vlaams Ministerie van Onderwijs en Vorming, 2010) – in which strategy flexibility is emphasized as a core goal – we think our results show that strategy variety and flexibility need to be emphasized more in the mathematics classroom. For example, teachers can operationalize this core goal by giving more attention to classroom discussions on “the how, the when, and the why” of strategy use (as in Selter, 1998). Through such discussions, children may be confronted with and start to try out new strategies by listening to the solution processes of their peers. Moreover, in those discussions teachers can push their students to reflect upon the suitability of a strategy for a given type of problem, and stimulate them to make connections to underlying mathematical principles (such as the addition/subtraction complement principle for the subtraction by addition strategy). Finally, such classroom discussions can also result in more positive attitudes, beliefs and emotions towards strategy variety and flexibility in mathematics education in general (see also Verschaffel, Luwel, et al., 2009).

9. Footnotes

1 People can also use a third strategy, the so-called indirect subtraction strategy in which they determine how much needs to be subtracted from the minuend to get to the subtrahend (e.g., 75 – 43 = . by 75 – 30 = 45 and 45 – 2 = 43; so the answer is 30 + 2 = 32) (De Corte & Verschaffel, 1987). This indirect subtraction strategy may be particularly efficient on problems with relatively large subtrahends (e.g., 81 – 79 = .). However, previous studies on people’s strategy use in subtraction revealed that participants use this strategy only very rarely or not even at all (Beishuizen, Van Putten, & Van Mulken, 1997; Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009b; Van Lieshout, 1997).
2 Several authors have reported the use of these addition-based strategies, but different terms are used to denote them, such as the forward strategy (Brissiaud, 1994), the adding-on-to strategy (Menne, 2001); solving subtractions by means of addition (Beishuizen, 1997), the short jump strategy (Blôte et al., 2000), the adding up strategy (Selter, 2001), or indirect addition (Torbeyns, Ghesquière, et al., 2009).

3 Although subject and context characteristics might also play an important role in strategy choice processes (e.g., Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009), we only included task characteristics to operationalize flexibility and adaptivity in this research project.

4 Children who learn to do math with the textbook “Vaardig en Vlot” (Lowagie & Staelens, 1998) are taught to climb the ladder (i.e., the name they use to describe subtraction by addition) for symbolically presented subtraction problems with minuends < 10 and the subtrahend larger than the difference (e.g., 9 − 5 = , 8 − 6 = , and 7 − 4 = ). When confronted with problems with minuends between 10 and 20, they continue to climb the ladder when the difference is small (e.g., 19 − 17 = , and 12 − 8 = ). When the minuend is between 20 and 100, children are taught to climb the ladder when the difference is smaller than 10 (e.g., 57 − 49 = . and 91 − 85 = .).

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**BRIEF BIOGRAPHY**

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