‘Εχεις μοι εἰπεῖν, ὦ Σίκρατες, ἃρα διδακτὸν ἢ ἁρετή; ἢ οὔ διδακτὸν ἀλλ’ ἄσκητον, ἢ οὔτε ἄσκητον οὔτε μαθητὸν, ἀλλ’ φύσει παραφηγοῖ τῶν ἀνθρώπων ἢ ἄλλω τινὶ τρόπω
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The scope of the MEJER is broad, both in terms of topics covered and disciplinary perspective, since the journal attempts to make connections between fields, theories, research methods, and scholarly discourses, and welcomes contributions on humanities, social sciences and sciences related to educational issues. It publishes original empirical and theoretical papers as well as reviews. Topical collections of articles appropriate to MEJER regularly appear as special issues (thematic issues).

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EDITOR'S INTRODUCTORY NOTE

INTRODUCTION TO SPECIAL ISSUE

Behaviour of students, teachers and future teachers in mental calculation and estimation

We are happy to present the first Special Issue of our new journal “MENON: Journal for Educational Research” which was introduced in 2012. Research in Mathematics Education is a significant area of educational research, which is included in the topics of this journal.

“Behaviour of students, teachers and future teachers in mental calculation and estimation” has been chosen as the subject for this special issue on the ground of a number of reasons which are presented below.

Over the past decades, many studies have been conducted in the field of mental calculations and estimation and more precisely in relation to the definition of these concepts, the identification of the strategies used by various age groups, the relationship with other concepts, such as number sense, the procedural and conceptual understanding among others.

Many educational systems have updated the teaching of numbers and operations in mathematics, incorporating mental calculations and estimations in their elementary and middle education curricula.

Nowadays, it is considered timely to conduct research in the implementation of the teaching of mental calculation and estimation with whole and rational numbers as well as the recording of students' behaviour and the training of pre-service and in-service teachers in these concepts.

During the last decade, researches on mental computation and estimation with rational numbers has been conducted in the Laboratory of “Nature and Life Mathematic” at the University of Western Macedonia, some of which are presented in this issue.

Most of the papers included in this issue, refer to mental calculations and estimations with rational numbers, a topic that is not very common in the literature and covers a wide age range including elementary school students, adults, as well as pre- and in-service teachers. The researches are presented according to the age range of the participants.

- In their study Greet, Bert, Torbeyns, Ghesquière and Verschaffel distinguish between two types of strategies for subtraction: (1) direct subtraction, and (2) subtraction by addition, and provide an overview of the results of 5 related studies using non-verbal methods to investigate the flexible use of these strategies in both single- and multi-digit subtraction. Adults, students and
elementary school students with mathematical learning disabilities have participated in this research.

- Anestakis and Lemonidis in their study, investigate the computational estimation ability of adult learners and implement a teaching intervention about computational estimation in a Junior High School for Adults. They suggest incorporating computational estimation into Second Chance Schools and into adult numeracy teaching practices in general.

- The two papers of Lemonidis, Nolka, Nikolantonakis and Lemonidis, Kaiafa examine the behaviour of 5th and 6th grade students in computational estimation and in mental calculations with rational numbers, respectively. In these studies, the relation between students’ performance in computational estimation and mental calculations with rational number and problem solving ability are also examined.

Four studies on this issue, refer to the prospective elementary teachers’ behaviour in mental calculation and estimation.

- Anestakis and Desli examined 113 prospective primary school teachers’ views of computational estimation and its teaching in primary school. Results revealed that the majority of prospective teachers identified the importance of computational estimation for both daily life and school.

- In their research Kourkoulos and Chalepaki interviewed and examined through a test 69 pre-service teachers aiming to investigate the factors that contribute to their computational estimation ability. They found five factors that contribute to computational estimation, such as the mathematical background and the attitude towards mathematics.

- Lemonidis, Tsakiridou, Panou and Griva used interviews to examine the knowledge and the strategy use of 50 pre-service teachers in multiplication tables and their mental flexibility in two-digit multiplications by using the method choice / no-choice by Lemaire & Siegler, (1995). The results showed that prospective teachers are not flexible in two-digit multiplications and their flexibility to mentally calculate two-digit multiplications is associated with their knowledge in prep and their prep response time.

- Koleza and Koleli have used a test to study the mental computations and estimation strategies of 87 pre-service teachers. The data revealed that the prospective teachers’ number sense concerning rational numbers, basic concepts of the decimal system and elementary numerical properties was very weak.

- Lemonidis, Mouratoglou and Pnevmatikos studied 80 in-service teachers’ performance and strategies in computational estimation and individual
differences concerning their age and work experience, their attitude towards mathematics and their prior performance in mathematics during high school years.

- The last paper of Lemonidis, Kermeli and Palaigeorgiou propose a teaching intervention to sixth grade students in order to promote understanding and enrich their conceptual strategy repertoire to carry out mental calculations with rational numbers. At the same time, three teachers’ attitudes towards teaching mental computation with rational numbers, were examined.

Finally, I would like to thank all the researchers from Belgium and Greece who contributed with their papers in this thematic issue, the colleagues from the laboratory of "Nature and Life Mathematics", the reviewers of the papers and Elias Indos for the organizational and technical support in the journal.

The Editor of the first Special Issue of “MENON: Journal for Educational Research”

Charalampos Lemonidis
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EXPLORING NUMBER SENSE IN SIXTH GRADE IN GREECE: AN INSTRUCTIONAL PROPOSAL AND ITS LEARNING RESULTS

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ABSTRACT

Surveys show that the majority of Greek students use procedural strategies in regards to mental calculations with rational numbers, i.e., they are just trying to apply rules without having the adequate understanding (Lemonidis, 2013, Lemonidis & Kaiafa in this issue). This study proposes a teaching intervention which aims to advance students understanding about mental calculations and enrich their conceptual strategies repertoire. For this purpose, three teachers of 6th grade were first trained and, later, asked to teach mental calculations operations with rational numbers for a period of three months. The post training understanding and skills of 66 students were compared with the answers of 462 5th and 6th grade students who participated in mathematics competition by using the same instrument (Lemonidis and Kaiafa in this issue). The results showed that students of the experimental group, exploited a larger pool of strategies and, especially, more conceptual strategies. It is interesting that teachers were not familiar enough with the strategies for executing mental operations with rational numbers. After the intervention, positive changes in their attitudes and knowledge were identified.

Keywords: number sense improvement, rational numbers, sixth grader’s strategies.

1. INTRODUCTION

The teaching and learning of number sense in elementary and middle years has been considered to be a significant topic in mathematics education (Anghileri, 2007; Lemonidis, 2013; NCTM, 2000; Verschaffel, Greer, & De Corte, 2007; Yang, 2005). Number sense plays a key role for understanding rational numbers and their operations. Overemphasis on written computation often hinders the children's mathematical thinking and comprehension (Anghileri, 2007; Reys & Yang, 1998).
However, related research studies show that children in elementary and middle grade levels are lacking number sense in rational numbers (Callingham, & Watson, 2004; Lemonidis, 2013; Markovits & Sowder, 1994; Reys & Yang, 1998; Yang, 2005). Therefore, teaching and learning number sense should be highlighted in elementary and middle school mathematics classrooms.

The teaching of rational numbers in Greece is concentrated more on rule-based calculations and less on in-depth understanding. In upper classes of elementary school, while the curriculum emphasizes the value of mental calculations, there are not concrete teaching proposals for mental computations with rational numbers. Our aim was to explore the effectiveness of a teaching intervention focused on mental calculations of rational numbers on teachers and students of sixth grade.

More specifically, the research questions in this paper were the following:

- Which were the pedagogical content knowledge and the practical skills of the teachers about mental calculations with rational numbers before and after the teaching intervention?
- How did the teaching intervention change the strategies with rational numbers of the sixth grade students?

2. BACKGROUND

Reys et al. (1999), in their review, identified six components of number sense:

1. Understanding of the meaning and size of numbers.
2. Understanding and use of equivalent representations of numbers.
3. Understanding the meaning and effect of operations.
4. Understanding and use of equivalent expressions.
5. Flexible computing and counting strategies for mental computation, written computation, and calculator use.
6. Measurement benchmarks (p. 62)

Yang and colleagues have coded subjects’ strategies as Number sense-based and Rule-based (Yang, 2003, 2005, 2007; Yang et al., 2009). Their criterion for distinguishing a strategy as a number sense-based was whether one or more components of number sense are evident in the person’s solution process (Yang, 2003, 2005, 2007). Some examples of number sense strategies are

a. the conversion of a fraction or a percentage to a decimal before operating on them,
b. the schematic representation of fractions (Caney and Watson, 2003)
c. residual thinking (Behr, et. al. 1984) where the fraction with the smaller residual is the bigger fraction.

On the other hand, rule-based strategies are based on memorized rules which are not necessarily linked to deep conceptual understanding.

Research has shown that many students in the upper and middle elementary grades are poor in number sense (Markovits & Sowder, 1994; McIntosh, et al. 1997;
Reys & Yang, 1998; Reys, et al. 1999; Van den Heuvel-Panhuizen, 1996, 2001; Yoshikawa, 1994; Verschaffel, Greer & DeCorte, 2007 Yang, 2005; Yang & Huang, 2004; Yang et al., 2008). The students are able to perform computations on familiar tasks by using school-learned procedures, but they perform poorly and employ rule-based methods on non-routine tasks.

Yang et al. (2009) have also identified teachers’ lack of number sense as a cause to students’ deficiencies in number sense: “Thus, children’s lack of number sense may be partly due to their teachers’ lack of number sense as well as not knowing how to help students develop number sense.” p. 383. Yang and colleagues have also argued that preservice teachers in Taiwan exhibit poor number sense, relying heavily on standard written algorithms (Yang, 2007; Yang et al., 2009). Although the preservice teachers were more capable of answering correctly number sense test items than their middle-school counterparts, the majority of their answers were also obtained by written computations based on standard algorithms.

However, number sense can be improved through instruction. Relevant studies with both students (Markovits and Sowder, 1994; Yang, 2002, 2003; Yang et al. 2004) and teachers (Kaminski, 2002; Whitacre, 2012, 2014; Whitacre and Nickerson, 2006) have demonstrated that number sense can be improved through instruction. For example, Yang (2003) reported on a semester-long (about 4 and a half months) quasi-experimental study of two fifth grade classes in Taiwan. Number sense activities were conducted in the experimental class, while the control class followed the standard mathematics curriculum. The authors used the Number Sense Rating Scale (Hsu et al., 2001) and found that the scores of the experimental class increased by 44% after instruction, while the scores of the control class increased only by 10%. Interviews showed that students from the experimental class used a higher proportion of number sense-based strategies in post-instruction and retention interviews. Whitacre (2012), in his literature review concluded that: “(1) number sense can be improved through instruction, and (2) there is more to be learned regarding how number sense improves with instruction.” p. 34.

3. METHODOLOGY

3.1. Research method

We used the framework of Ball and colleagues’ for examining teachers’ mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008). In their framework, the authors discern the knowledge that teachers apply in the classroom in two domains: subject matter and pedagogical content knowledge. Subject Matter Knowledge includes three subcategories: 1) Common Content Knowledge which refers to the general knowledge of mathematics not specific to teaching. 2) Specialized Content Knowledge which refers to the mathematical knowledge required for the specific learning content e.g. the knowledge teachers need to explain patterns of students’ errors or decide whether a nonstandard approach may work. 3) The Horizon Content
Knowledge which refers to the “awareness of how Mathematical topics are related over the span of mathematics included in the curriculum” (p. 403). Pedagogical Content Knowledge includes three subcategories: 1) Knowledge of Content and Students which combines knowing about students and knowing about mathematics. 2) Knowledge of Content and Teaching which concentrates on the design of instruction for the specific content. 3) Knowledge of Content and Curriculum which refers to the acknowledgement of the ways mathematics are developed in a teacher’s curriculum.

In our research, we examined the subject matter and pedagogical content knowledge of the three teachers, more specifically, we examined their Specialized Content Knowledge, Knowledge of Content and Students, and Knowledge of Content and Teaching.

Teachers attended a training program and then applied the newly acquired skills and knowledge in their classrooms. An action research approach was used to explore teachers’ subject matter and pedagogical content knowledge toward mental calculations with rational numbers before and after the intervention.

### 3.2. Participants

The participants in this study were three teachers of 6th grade and their 66 students. Teachers in this study were identified through a process of convenience sampling. Three teachers of 6th grade responded that they were interested in participating in the study. The teachers taught for the first time, in sixth grade.

### 3.3. Research Design

The study was conducted in five stages.

In the first stage, the authors interviewed 3 teachers of 6th grade. The aim of the interview was to explore their prior subject matter and pedagogical content knowledge for mental calculations on rational numbers.

In the second stage, a training session was conducted with each teacher separately. The training session lasted for two hours, and teachers were also given a 27-page booklet containing additional training material. The training took place before the Christmas holidays so as the teachers could have enough time to study the educational material, look at the various resources and organize their instructional design.

In the third stage, teachers used their instructional plans in the classroom. This stage lasted for 3 months.

In the fourth stage, the researchers conducted again personal interviews with each teacher in order to explore changes in their subject matter and pedagogical content knowledge. In those interviews, teachers also gave their views on the learning results of their new teaching approach.

Finally, in the fifth stage, questionnaires containing four problems of mental calculations with rational numbers were given to the 66 students. The problems were
identical to the ones given to 462 students of fifth and sixth grade who participated in a contest of mathematics. These students were not taught mental calculation strategies with rational numbers and, thus, the number sense strategies they used were spontaneous and self-developed. The comparison of these two groups of students could inform us about the effectiveness of the proposed teaching intervention.

3.4. Key features of teachers’ individual training session

The training session focused on the following topics:

- *The concept of mental calculations and their difference to written algorithms.*
- The aim was to clarify the meaning of mental calculations and their relationship with written calculation algorithms.
- *The value of mental calculations with rational numbers in number sense and operations.*
- The trainers emphasized that mental calculations of rational numbers improve the understanding of the operations and rational numbers in contrast to what happens with written algorithms where the operations can be executed without understanding.
- *Students’ strategies and errors*
- Teachers should be familiar with the various strategies that students exploit when performing operations with rational numbers. An analytic presentation of students’ systematic errors, difficulties and misconceptions, according to research data was given.
- *Instructional methods and metacognitive processes*
- The importance of using metacognitive processes during the teaching of mental calculations was highlighted, and teachers were asked, for example, to make students talk and explain how they thought. The significance of presenting and sharing all students’ strategies was also indicated as a mean to enrich the strategies employed by students.

3.5. Data collection

Two types of data were collected:

- a. semi-structured interviews before and after the training session and the instruction and
- b. a questionnaire with 4 problems on rational numbers, which was administered to students in the end of the corresponding instruction.

3.6. Interviews

Before the intervention, the 3 teachers were asked about the importance of teaching mental calculations and whether it should follow or precede the teaching of rule-based calculations, about how they teach mental computations and what resources
they use and whether the majority of students in a typical classroom is able to develop and apply mental calculations when solving problems with rational numbers. The teachers were also asked to indicate the strategies that students use (in general and on specific problems) and the errors that students make on fractions, decimals and percentages.

After the intervention, the teachers were asked whether and how the training session together with the teaching intervention changed their initial perceptions about the importance of mental calculations, how they changed the way that they taught mental calculations, how students’ behavior changed and if they were satisfied with the initial training session.

3.7 Questionnaire

The four questions, that students of both groups had to answer, were:

Q1: I calculate with my mind: 1- ¼. I use two ways to answer. Every time, I write the way I thought
Q2: I calculate with my mind: 1/2:1/4. I use two ways to answer. Every time, I write the way I thought.
Q3: I compare the fractions 3/7 and 5/8. Which is greater? I use two ways to answer. Every time, I write the way I thought.
Q4: I find the 90% of 40. I use two ways to answer. Every time, I write the way I thought.

4. RESULTS

4.1. Teachers’ subject matter and pedagogical content knowledge

4.1.1 Before the intervention

All three teachers considered significant the teaching of mental calculations and pinpointed a number of advantages. More specifically, teacher 1 emphasized the conceptual understanding that results from practicing with mental calculations, teacher 2 focused on the usefulness of mental calculations with rational numbers in everyday life, while teacher 3 noted the effect of mental calculations in connecting mathematics with everyday life and the advantages of releasing students from the rule-based algorithms.

The teachers had different views on the importance of teaching mental calculations and whether it should follow or precede the teaching of rule-based calculations. Teacher 1 considered mental calculations as more significant and argued that teaching of mental calculations should precede the teaching of rule-based thinking. Teacher 2 also supported the same order, but she didn’t select one of the two as more significant. Teacher 3 argued that teaching of mental and written calculations was of equal importance, however, she proposed the introduction of written calculations first and the presentation of mental calculations later: "First you need the theory and after having understand it, we can show them the clever ways".
Teacher 3, considered written algorithms as the "theory" and mental calculations as merely some "smart ways" of calculation.

The teachers took advantage of mental calculations in different ways. The teacher 1 said that she devotes most of her time to mental calculations and that she tries to visualize the problems. Her goal was to focus more on understanding and less on algorithms. Teacher 2 said that she teaches mental calculations once a week with her own examples which are not contained in the textbook. The teacher 3, claimed that she teaches mental calculations when she gives explanations with small numbers on rule-based algorithms. All three teachers used a variety of resources (from textbooks to internet resources) and examples (from every day to school life). Interestingly, all three teachers said that they do not use computers because of logistical and technical difficulties, although they argued that they do appreciate the usefulness of technology.

Teachers were asked to assess whether the majority of students in a typical classroom is able to apply mental calculations when solving problems with rational numbers. According to their answers, they considered mental calculations as a difficult topic and believed that only the best students could develop such skills.

Teachers were also asked to indicate the strategies that students use on fractions, decimals and percentages. Teacher 1 stated that she did not remember any specific strategy, the other two said that students work on decimals in a similar way to the one of the whole numbers. Teacher 2 also argued that: "In fractions, my students calculate the difference between the numerator and the denominator." Teacher 3 also stated that students convert opposite fractions to decimals in order to operate on them. From the teachers’ responses, it was quite obvious that they did not know much about their students’ strategies when operating with rational numbers.

When asked about the common errors that students make with fractions, decimals and percentages, the three teachers identified different issues and each reported only a small number of problems and misconceptions. Teacher 1 reported that students often do not convert opposite fractions to the same denominator and they add the numerators and the denominators. For decimal numbers, they said that students do not pay attention to the place value. Teacher 1 also pointed out that there is a difficulty when dividing decimals or fractions. Teacher 2 noted the following two errors: Students think that the unit fraction is the number 1 and they believe that 0.4 and 0.40 are different. Students also consider that between two decimal numbers such as 1.4 and 1.5 no other numbers exist. Teacher 3 indicated that there is a general difficulty in understanding fractions and that student’s think that the multiplication of numbers with decimals results to smaller numbers.

To examine whether teachers were familiar with their students’ mental strategies, we asked them the following question:

“Which are the strategies that students would use if they were asked to estimate mentally the following operations: a) 1-1 / 4, b) ½: 1/4, c) compare the fractions 3/7 and 5/8 and d) find 90% of 40”
For the first question the strategies proposed were:
1. Rule-based: convert 1 to 4 quarters and remove one quarter.
2. Representational: Consider a geometric figure with four parts from which they remove \( \frac{1}{4} \).
3. Convert the whole number in the base of 100. Students will think of 1 as 100, divide 100 into 4 parts and remove thereby one part.

For this first question teacher 1 replied the strategies 1 and 2, teacher 2 the strategies 2 and 3 and teacher 3 only the strategy 1.

Teacher 1 said that the second problem was very difficult and that students might have solved it only by applying rule-based thinking. Teacher 2 said that students "would have divided one by two and then divided the result with two". Teacher 3 said: "I do not know, I need to solve it first by myself." Hence, the first teacher proposed a rule-based solution while the other two seemed unable to solve the problem by themselves.

For the comparison of fractions \( \frac{3}{7} \) and \( \frac{5}{8} \), teachers proposed the following strategies:
1. Convert the two fractions to same denominator.
2. Convert the two fractions to decimals.
3. Compare the two fractions with the benchmark of \( \frac{1}{2} \).

In this question, teacher 1 replied the strategies 1 and 3, teacher 2 named a wrong criterion ("3 of the 7 is more than 5 of 8"), and teacher 3 the strategies 1 and 2.

In the fourth question, teacher 1 did not suggest any strategy, teacher 2 replied that students would have multiplied mentally the numbers \( (4 \times 9) \) and teacher 3 replied that she didn’t know because she hadn’t taught about percentages yet.

Hence, we can conclude that teachers did not know the variety of strategies exploited by students, but they also demonstrated a lack of content knowledge since they were not able to answer some of the questions.

### 4.1.2. After the intervention

When asked if the training session together with the teaching intervention changed their initial perceptions about the importance of mental calculations, teachers 1 and 3 said that they had already appreciated the value of mental calculations. Only teacher 3 noted that "after the training, I saw practically that mental calculations, can help students." Teacher 2 argued that the whole activity helped her revise some initial views as mental calculations seemed to “sharpen students’ minds”. The teachers stated that they were satisfied with the initial training session.

Teachers suggested that all students were actively involved in their teaching while the weak students needed more time in order to familiarize with mental computations. Teacher 1 stated: "Initially, the mental computations with rational numbers intrigued the best students. However, later, the others and even some weaker students become more engaged. As they started to understand more the
strategies, they felt more confident about their answers and participated more actively in the course. The three teachers argued that all of their students participated in the classroom activities, even the weakest. Teacher 2 added: "Some students understood rational numbers through mental calculations and afterwards the written operations seemed as simple processes for them. Others started to understand better when the calculations were written. When the examples were simple and similar to the ones I used, students used mental computation with success. When the examples became more complex, students preferred written algorithms." Teachers also noted that many students knew and used appropriate calculation strategies for different problems.

4.2. Students’ behaviors after interventional instruction

4.2.1 Strategies usage

The following table shows the percentages of rule-based and number sense strategies that were used by the students (group A) who participated in the study and students (group B) who participated in the competition.

<table>
<thead>
<tr>
<th>STRATEGY</th>
<th>Q1:1-1/4</th>
<th>Q2:3/4:1/4</th>
<th>Q3:3/7&amp;5/8</th>
<th>Q4:90%of 40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group A</td>
<td>25 (39.7%)</td>
<td>31 (52.5%)</td>
<td>17 (35.4%)</td>
<td>33 (54%)</td>
</tr>
<tr>
<td>Group B</td>
<td>180 (67%)</td>
<td>185 (80.5%)</td>
<td>122 (58%)</td>
<td>140 (62.5%)</td>
</tr>
<tr>
<td>Rule-based strategies</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number sense strategies</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Group A</td>
<td>38 (60.3%)</td>
<td>28 (47.5%)</td>
<td>31 (64.6%)</td>
<td>28 (45.9%)</td>
</tr>
<tr>
<td>Group B</td>
<td>89 (33%)</td>
<td>45 (19.5%)</td>
<td>88 (41.9%)</td>
<td>84 (37.5%)</td>
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<td>z=4</td>
<td>p&lt;0.001</td>
<td>p&lt;0.001</td>
<td>p=2.84</td>
<td>p=1.19</td>
</tr>
<tr>
<td>p&lt;0.001</td>
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</tbody>
</table>

The students of group A used a smaller percentage of rule-based strategies and a larger percentage of number sense strategies in comparison to the students of group B. The 2-sample z-test validated this observation by indicating that in three (Q1, Q2 and Q3) of the four questions, the difference was significant. The proposed instruction promoted number sense strategies when operating on rational numbers. Students of group A understood better the operations with rational numbers despite the fact that the students who participated in the contest probably had more positive attitudes towards mathematics.

Another effect was that the respective students used a greater variety of number sense strategies. According to Table 2, in the questions Q1, Q2 and Q4 the students of the competition used 2, 2 and 3 number sense strategies respectively, while the students of group A used 3, 4 and 4 number sense strategies.
Table 2: Types of number sense strategies used by students

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<tbody>
<tr>
<td>Number sense strategies</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Group A</td>
<td>38 (60.3%)</td>
<td>28 (47.5%)</td>
<td>31 (64.6%)</td>
<td>28 (45.9%)</td>
</tr>
<tr>
<td>Group B</td>
<td>89 (33%)</td>
<td>45 (19.5%)</td>
<td>88 (41.9%)</td>
<td>84 (37.5%)</td>
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<tr>
<td>Converting a Fraction or a Percent to a Decimal</td>
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<tr>
<td>Group A</td>
<td>23 (36.5%)</td>
<td>17 (29%)</td>
<td>11 (23%)</td>
<td>7 (11.5%)</td>
</tr>
<tr>
<td>Group B</td>
<td>82 (30.5%)</td>
<td>38 (16.5%)</td>
<td>56 (26.5%)</td>
<td>42 (18.75%)</td>
</tr>
<tr>
<td>Representation</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Group A</td>
<td>9 (14.3%)</td>
<td>7 (12%)</td>
<td>12 (25%)</td>
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</tr>
<tr>
<td>Group B</td>
<td>7 (2.5%)</td>
<td>18 (8.5%)</td>
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<td></td>
</tr>
<tr>
<td>Mixed representation</td>
<td></td>
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<tr>
<td>Converting to a decimal</td>
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</tr>
<tr>
<td>Group A</td>
<td>6 (9.5%)</td>
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<td></td>
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<tr>
<td>Group B</td>
<td></td>
<td></td>
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<tr>
<td>Benchmarks to ½ or 10%</td>
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<tr>
<td>Group A</td>
<td>2 (3.4%)</td>
<td>6 (12.5%)</td>
<td>3 (5%)</td>
<td></td>
</tr>
<tr>
<td>Group B</td>
<td>7 (3%)</td>
<td>7 (3.5%)</td>
<td>19 (8.5%)</td>
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</tr>
<tr>
<td>Mixed representation</td>
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<tr>
<td>Benchmarks to ½</td>
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</tr>
<tr>
<td>Group A</td>
<td>2 (3.4%)</td>
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<tr>
<td>Group B</td>
<td></td>
<td></td>
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<tr>
<td>Residual thinking</td>
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<tr>
<td>Group A</td>
<td>2 (4%)</td>
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<tr>
<td>Group B</td>
<td>7 (3.5%)</td>
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<tr>
<td>Benchmarks: reduction in unit</td>
<td></td>
<td></td>
<td></td>
<td>12 (19.5%)</td>
</tr>
<tr>
<td>Group A</td>
<td></td>
<td></td>
<td></td>
<td>23 (10.3%)</td>
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<tr>
<td>Group B</td>
<td></td>
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</tr>
<tr>
<td>Benchmarks: 10% of 40=4</td>
<td></td>
<td></td>
<td></td>
<td>6 (10%)</td>
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<tr>
<td>Group A</td>
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</tr>
<tr>
<td>Group B</td>
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</table>

5. DISCUSSION

The initial interview with the teachers revealed that they confronted serious shortcomings in the content knowledge of mental calculations with rational numbers, what Ball and colleagues (2008) named as Common Content Knowledge. It was also obvious that they were not aware of the knowledge, attitudes and misconceptions of students in the specific subject matter, what Ball and colleagues (2008) named as Specialized Content Knowledge. The lack of knowledge on the mathematical content
of mental calculations with rational numbers have also been noted in other studies (Post et al. 1988; Cramer & Lesh 1988; Khoury & Zazkis, 1994). Hence, it is necessary for teacher education programs to enhance Common Content Knowledge and Specialized Content Knowledge on mental calculations with rational numbers.

The teaching intervention improved students' knowledge of number sense strategies as indicated by the interviews and the questionnaires. Students who belonged to the experimental group used number sense strategies in comparison with the students that participated in the competition. We have to note that the students of the second group participated in the competition by their own willingness, and hence, we can claim that this group was a challenging one for comparison since, at least those students had a more advanced interest in mathematics.

A limitation of this research is the lack of systematic monitoring of teachers' instructional planning and decisions during the three month teaching period. Although, the teachers described in the final interviews their approaches, a more analytic recording of their practices would probably reveal more interesting issues about each class separately. Future research could also exploit the theory of mathematics teacher noticing (Jacobs, Lamb, & Philipp, 2010; Sherin, Jacobs & Philipp, 2011) in order to examine changes in teachers decisions and knowledge.

REFERENCES


**BRIEF BIOGRAPHIES**

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